1.7 Introduction to Continuity

This section introduces the idea of *continuity* on an interval and at a point. This leads to the concept of limit, which is investigated in the next section.

Continuity of a Function on an Interval: Graphical Viewpoint

Roughly speaking, a function is said to be *continuous* on an interval if its graph has no breaks, jumps, or holes in that interval. Continuity is important because, as we shall see, continuous functions have many desirable properties.

For example, to locate the zeros of a function, we often look for intervals where the function changes sign. In the case of the function $f(x) = 3x^3 - x^2 + 2x - 1$, for instance, we expect[11] to find a zero between 0 and 1 because $f(0) = -1$ and $f(1) = 3$. (See Figure 1.75.) To be sure that $f(x)$ has a zero there, we need to know that the graph of the function has no breaks or jumps in it. Otherwise the graph could jump across the x-axis, changing sign but not creating a zero. For example, $f(x) = 1/x$ has opposite signs at $x = -1$ and $x = 1$, but no zeros for $-1 \le x \le 1$ because of the break at $x = 0$. (See Figure 1.76.) To be certain that a function has a zero in an interval on which it changes sign, we need to know that the function is defined and continuous in that interval.

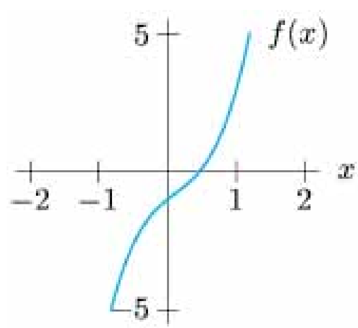

Figure 1.75 The graph of $f(x) = 3x^3 - x^2 + 2x - 1$

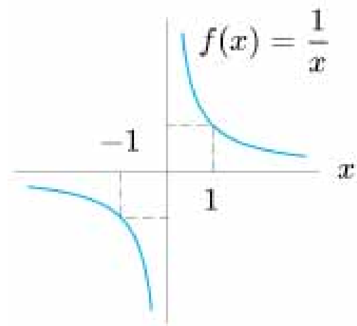

Figure 1.76 No zero although $f(-1)$ and $f(1)$ have opposite signs

A continuous function has a graph which can be drawn without lifting the pencil from the paper.

Example: The function $f(x) = 3x^3 - x^2 + 2x - 1$ is continuous on any interval. (See Figure 1.75.)

Example: The function $f(x) = 1/x$ is not defined at $x = 0$. It is continuous on any interval not containing the origin. (See Figure 1.76.)

Example: Suppose $p(x)$ is the price of mailing a first-class letter weighing x ounces. It costs 41¢ for one ounce or less, 58¢ between the first and second ounces, and so on. So the graph (in Figure 1.77) is a series of steps. This function is not continuous on any open interval containing a positive integer because the graph jumps at these points.

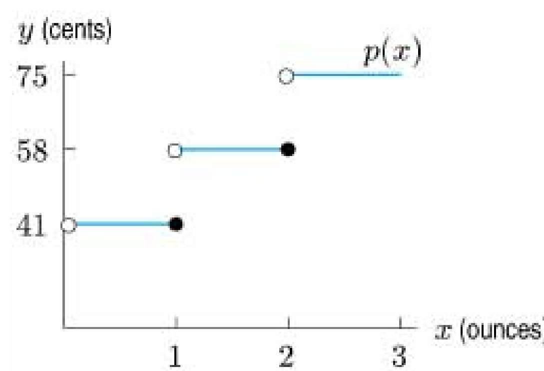

Figure 1.77 Cost of mailing a letter

Which Functions are Continuous?

Requiring a function to be continuous on an interval is not asking very much, as any function whose graph is an unbroken curve over the interval is continuous. For example, exponential functions, polynomials, and the sine and cosine are continuous on every interval. Rational functions are continuous on any interval in which their denominators are not zero. Functions created by adding, multiplying, or composing continuous functions are also continuous.

The Intermediate Value Theorem

Continuity tells us about the values taken by a function. In particular, a continuous function cannot skip values. For example, the function in the next example must have a zero because its graph cannot skip over the x-axis.

Example 1

What do the values in Table 1.18 tell you about the zeros of $f(x) = \cos x - 2x^2$?

Table 1.18

x	$f(x)$
0	1.00
0.2	0.90
0.4	0.60
0.6	0.11
0.8	-0.58
1.0	-1.46

Solution

Since $f(x)$ is the difference of two continuous functions, it is continuous. We conclude that $f(x)$ has at least one zero in the interval $0.6 < x < 0.8$, since $f(x)$ changes from positive to negative on that interval. The graph of $f(x)$ in Figure 1.78 suggests that there is only one zero in the interval $0 \leq x \leq 1$, but we cannot be sure of this from the graph or the table of values.

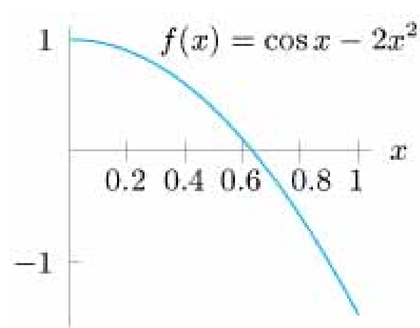

Figure 1.78 Zeros occur where the graph of a continuous function crosses the horizontal axis

In the previous example, we concluded that $f(x) = \cos x - 2x^2$ has a zero between $x = 0$ and $x = 1$ because $f(x)$ is positive at $x = 0$ and negative at $x = 1$. More generally, an intuitive notion of continuity tells us that, as we follow the graph of a continuous function f from some point $(a, f(a))$ to another point $(b, f(b))$, then f takes on all intermediate values between $f(a)$ and $f(b)$. (See Figure 1.79.) This is:

Theorem 1.1: Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

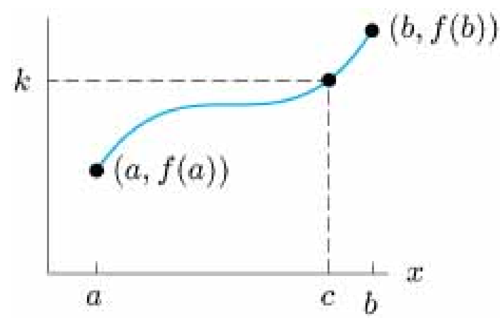

Figure 1.79 The Intermediate Value Theorem

The Intermediate Value Theorem depends on the formal definition of continuity given in Section 1.8. See also www.wiley.com/college/hugheshallett. The key idea is to find successively smaller subintervals of $[a, b]$ on which f changes from less than k to more than k. These subintervals converge on the number c.

Continuity of a Function at a Point: Numerical Viewpoint

A function is continuous if nearby values of the independent variable give nearby values of the function. In practical work, continuity is important because it means that small errors in the independent variable lead to small errors in the value of the function.

Example: Suppose that $f(x) = x^2$ and that we want to compute $f(\pi)$. Knowing f is continuous tells us that taking $x = 3.14$ should give a good approximation to $f(\pi)$, and that we can get as accurate an approximation to $f(\pi)$ as we want by using enough decimals of π.

Example: If $p(x)$ is the cost of mailing a letter weighing x ounces, then $p(0.99) = p(1) = 41$¢, whereas $p(1.01) = 58$¢, because as soon as we get over 1 ounce, the price jumps up to 58¢. So a small difference in the weight of a letter can lead to a significant difference in its mailing cost.

Hence p is not continuous at $x = 1$.

In other words, if $f(x)$ is continuous at $x = c$, the values of $f(x)$ approach $f(c)$ as x approaches c. In Section 1.8, we discuss the concept of a *limit*, which allows us to define more precisely what it means for the values of $f(x)$ to approach $f(c)$ as x approaches c.

Example 2

Investigate the continuity of $f(x) = x^2$ at $x = 2$.

Solution

From Table 1.19, it appears that the values of $f(x) = x^2$ approach $f(2) = 4$ as x approaches 2. Thus f appears to be continuous at $x = 2$. Continuity at a point describes behavior of a function *near* a point, as well as *at* the point.

Table 1.19 *Values of x^2 near $x = 2$*

x	1.9	1.99	1.999	2.001	2.01	2.1
x^2	3.61	3.96	3.996	4.004	4.04	4.41

Exercises and Problems for Section 1.7

Exercises

Are the functions in Exercises 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 continuous on the given intervals?

1. $2x + x^{2/3}$ on [-1, 1]

2. $2x + x^{-1}$ on [-1, 1]

3. $\frac{1}{x-2}$ on [-1, 1]

4. $\frac{1}{x-2}$ on [0, 3]

5. $\frac{1}{\sqrt{2x-5}}$ on [3, 4]

6. $\frac{x}{x^2+2}$ on [-2, 2]

7. $\frac{1}{\cos x}$ on $[0, \pi]$

8. $\frac{1}{\sin x}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

9. $\dfrac{e^x}{e^x - 1}$ on [-1, 1]

10. $\dfrac{e^{\sin\theta}}{\cos\theta}$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$

In Exercises 11, 12, 13 and 14, show that there is a number c, with $0 \le c \le 1$, such that $f(c) = 0$.

11. $f(x) = x^3 + x^2 - 1$

12. $f(x) = e^x - 3x$

13. $f(x) = x - \cos x$

14. $f(x) = 2^x - 1/x$

15. Are the following functions continuous? Explain.

(a) $f(x) = \begin{cases} x & x \le 1 \\ x^2 & 1 < x \end{cases}$

(b) $g(x) = \begin{cases} x & x \le 3 \\ x^2 & 3 < x \end{cases}$

Problems

16. Which of the following are continuous functions of time?

(a) The quantity of gas in the tank of a car on a journey between New York and Boston.

(b) The number of students enrolled in a class during a semester.

(c) The age of the oldest person alive.

17. An electrical circuit switches instantaneously from a 6 volt battery to a 12 volt battery 7 seconds after being turned on. Graph the battery voltage against time. Give formulas for the function represented by your graph. What can you say about the continuity of this function?

18. A car is coasting down a hill at a constant speed. A truck collides with the rear of the car, causing it to lurch ahead. Graph the car's speed from a time shortly before impact to a time shortly after impact. Graph the distance from the top of the hill for this time period. What can you say about the continuity of each of these functions?

19. Find k so that the following function is continuous on any interval:

$$f(x) = \begin{cases} kx & x \le 3 \\ 5 & 3 < x \end{cases}$$

20. Find k so that the following function is continuous on any interval:

$$f(x) = \begin{cases} kx & 0 \le x < 2 \\ 3x^2 & 2 \le x \end{cases}$$

21. Is the following function continuous on [-1, 1]?

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

22. If possible, choose k so that the following function is continuous on any interval:

$$f(x) = \begin{cases} \frac{5x^3 - 10x^2}{x - 2} & x \neq 2 \\ k & x = 2 \end{cases}$$

23. Find k so that the following function is continuous on any interval:

$$j(x) = \begin{cases} k \cos x & x \leq 0 \\ e^x - k & x > 0 \end{cases}$$

24. Discuss the continuity of the function g graphed in Figure 1.80 and defined as follows:

$$g(\theta) = \begin{cases} \frac{\sin \theta}{\theta} & \text{for } \theta \neq 0 \\ 1/2 & \text{for } \theta = 0. \end{cases}$$

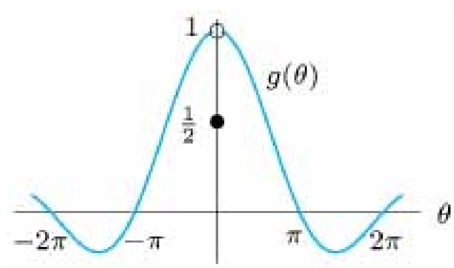

Figure 1.80

25.
(a) What does a graph of $y = e^x$ and $y = 4 - x^2$ tell you about the solutions to the equation $e^x = 4 - x^2$?

(b) Evaluate $f(x) = e^x + x^2 - 4$ at $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$. In which intervals do the solutions to $e^x = 4 - x^2$ lie?

26. Let $p(x)$ be a cubic polynomial with $p(5) < 0$, $p(10) > 0$, and $p(12) < 0$. What can you say about the number and location of zeros of $p(x)$?

27. Sketch the graphs of three different functions that are continuous on $0 \leq x \leq 1$ and that have the values given in the table. The first function is to have exactly one zero in [0, 1], the second is to have at least two zeros in the interval [0.6, 0.8], and the third is to have at least two zeros in the interval [0, 0.6].

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	0.90	0.60	0.11	-0.58	-1.46

28. (a) Sketch the graph of a continuous function f with *all* of the following properties:

(i) $f(0) = 2$

(ii) $f(x)$ is decreasing for $0 \le x \le 3$

(iii) $f(x)$ is increasing for $3 < x \le 5$

(iv) $f(x)$ is decreasing for $x > 5$

(v) $f(x) \to 9$ as $x \to \infty$

(b) Is it possible that the graph of f is concave down for all $x > 6$? Explain.

29. A 0.6 ml dose of a drug is injected into a patient steadily for half a second. At the end of this time, the quantity, Q, of the drug in the body starts to decay exponentially at a continuous rate of 0.2% per second. Using formulas, express Q as a continuous function of time, t in seconds.

30. Use a computer or calculator to sketch the functions

$$y(x) = \sin x \quad \text{and} \quad z_k(x) = \mathrm{k}e^{-x}$$

for k = 1,2,4,6,8,10. In each case find the smallest positive solution of the equation $y(x) = z_k(x)$. Now define a new function f by

$$f(k) = \{\text{Smallest positive solution of } y(x) = z_k(x)\}.$$

Explain why the function $f(k)$ is not continuous on the interval $0 \le k \le 10$.