CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Vectors**
  Geometric definition of vector addition, subtraction and scalar multiplication, resolving into $\vec{i}$, $\vec{j}$, and $\vec{k}$ components, magnitude of a vector, algebraic properties of addition and scalar multiplication.
- **Dot Product**
  Geometric and algebraic definition, algebraic properties, using dot products to find angles and determine perpendicularity, the equation of a plane with given normal vector passing through a given point, projection of a vector in a direction given by a unit vector.
- **Cross Product**
  Geometric and algebraic definition, algebraic properties, cross product and volume, finding the equation of a plane through three points.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER THIRTEEN

Exercises

In Exercises 1–2, is the quantity a vector or a scalar? Compute it.

1. $\vec{u}\cdot\vec{v}$, where $\vec{u}=2\vec{i}-3\vec{j}-4\vec{k}$ and $\vec{v}=\vec{k}-\vec{j}$

2. $\vec{u}\times\vec{v}$, where $\vec{u}=2\vec{i}-3\vec{j}-4\vec{k}$ and $\vec{v}=3\vec{i}-\vec{j}+\vec{k}$.

3. Resolve the vectors in Figure 13.45 into components.

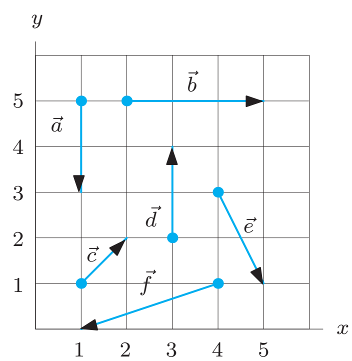

Figure 13.45

4. Resolve vector $\vec{v}$ into components if $\|\vec{v}\|=8$ and the direction of $\vec{v}$ is shown in Figure 13.46.

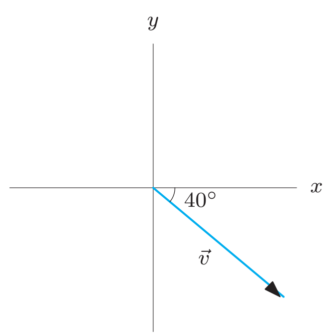

Figure 13.46

For Exercises 5–7, perform the indicated operations on the following vectors:

$$\vec{c}=\vec{i}+6\vec{j},\quad \vec{x}=-2\vec{i}+9\vec{j},\quad \vec{y}=4\vec{i}-7\vec{j}.$$

5. $5\vec{c}$

6. $\vec{c}+\vec{x}+\vec{y}$

7. $\|\vec{x}-\vec{c}\|$

In Exercises 8–17, use $\vec{v}=2\vec{i}+3\vec{j}-\vec{k}$ and $\vec{w}=\vec{i}-\vec{j}+2\vec{k}$ to calculate the given quantities.

8. $\vec{v}+2\vec{w}$

9. $3\vec{v}-\vec{w}-\vec{v}$

10. $\|\vec{v}+\vec{w}\|$

11. $\vec{v}\cdot\vec{w}$

12. $\vec{v}\times\vec{w}$

13. $\vec{v}\times\vec{v}$

14. $(\vec{v}\cdot\vec{w})\vec{v}$

15. $(\vec{v}\times\vec{w})\cdot\vec{w}$

16. $(\vec{v}\times\vec{w})\times\vec{w}$

17. $(\vec{v}\times\vec{w})\times(\vec{v}\times\vec{w})$

In Exercises 18–19, find a normal vector to the plane.

18. $2x+y-z=5$

19. $2(x-z)=3(x+y)$

20. Find the equation of the plane through the origin which is parallel to $z=4x-3y+8$.

21. Let $\vec{v}=3\vec{i}+2\vec{j}-2\vec{k}$ and $\vec{w}=4\vec{i}-3\vec{j}+\vec{k}$. Find each of the following:

(a) $\vec{v}\cdot\vec{w}$
(b) $\vec{v}\times\vec{w}$
(c) A vector of length 5 parallel to vector $\vec{v}$
(d) The angle between vectors $\vec{v}$ and $\vec{w}$
(e) The component of $\vec{v}$ in the direction of $\vec{w}$
(f) A vector perpendicular to vector $\vec{v}$
(g) A vector perpendicular to both vectors $\vec{v}$ and $\vec{w}$

In Exercises 22–27, find a vector with the given property.

22. Length 10, parallel to $2\vec{i}+3\vec{j}-\vec{k}$.

23. Unit vector perpendicular to $\vec{i}+\vec{j}$ and $\vec{i}-\vec{j}-\vec{k}$

24. Unit vector in the xy-plane perpendicular to $3\vec{i}-2\vec{j}$.

25. A vector normal to $4(x-1)+6(z+3)=12$.

26. The vector obtained from $4\vec{i}+3\vec{j}$ by rotating it 90° counterclockwise.

27. A nonzero vector perpendicular to $\vec{v}=3\vec{i}-\vec{j}+\vec{k}$ and $\vec{w}=\vec{i}-2\vec{j}+\vec{k}$.

28. Which of the following vectors are parallel?

$$\vec{u}=2\vec{i}+4\vec{j}-2\vec{k}, \quad \vec{p}=\vec{i}+\vec{j}+\vec{k},$$
$$\vec{v}=\vec{i}-\vec{j}+3\vec{k}, \quad \vec{q}=4\vec{i}-4\vec{j}+12\vec{k},$$
$$\vec{w}=-\vec{i}-2\vec{j}+\vec{k}, \quad \vec{r}=\vec{i}-\vec{j}+\vec{k}.$$

In Exercises 29–34, find the parallel and perpendicular components of the force vector $\vec{F}$ in the direction of the displacement vector $\vec{d}$. Then find the work W done by $\vec{F}$ though the displacement $\vec{d}$.

29. $\vec{F}=2\vec{i}+4\vec{j}, \quad \vec{d}=\vec{i}+2\vec{j}$

30. $\vec{F}=-2\vec{i}-4\vec{j}, \quad \vec{d}=\vec{i}+2\vec{j}$

31. $\vec{F}=2\vec{i}+4\vec{j}, \quad \vec{d}=2\vec{i}-1\vec{j}$

32. $\vec{F}=2\vec{i}+4\vec{j}, \quad \vec{d}=3\vec{i}-4\vec{j}$

33. $\vec{F}=2\vec{i}, \quad \vec{d}=\vec{i}+\vec{j}$

34. $\vec{F}=5\vec{i}+2\vec{j}, \quad \vec{d}=3\vec{j}$

Problems

35. Figure 13.47 shows a rectangular box containing several vectors. Are the following statements true or false? Explain.

(a) $\vec{c}=\vec{f}$ **(b)** $\vec{a}=\vec{d}$ **(c)** $\vec{a}=-\vec{b}$
(d) $\vec{g}=\vec{f}+\vec{a}$ **(e)** $\vec{e}=\vec{a}-\vec{b}$ **(f)** $\vec{d}=\vec{g}-\vec{c}$

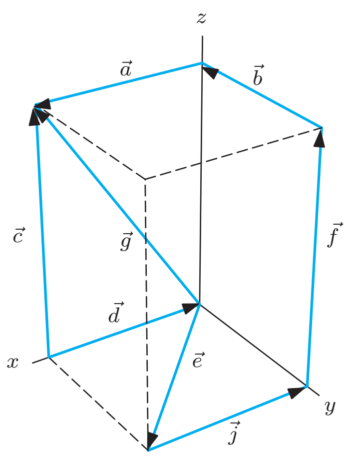

Figure 13.47

36. Shortly after takeoff, a plane is climbing northwest through still air at an airspeed of 200 km/hr, and rising at a rate of 300 m/min. Resolve its velocity vector into components. The x-axis points east, the y-axis points north, and the z-axis points up.

37. A boat is heading due east at 25 km/hr (relative to the water). The current is moving toward the southwest at 10 km/hr.

(a) Give the vector representing the actual movement of the boat.

(b) How fast is the boat going, relative to the ground?

(c) By what angle does the current push the boat off of its due east course?

38. A model rocket is shot into the air at an angle with the earth of about 60°. The rocket is going fast initially but slows down as it reaches its highest point. It picks up speed again as it falls to earth.

(a) Sketch a graph showing the path of the rocket. Draw several velocity vectors on your graph.

(b) A second rocket has a parachute that deploys as it begins its descent. How do the velocity vectors from part (a) change for this rocket?

39. A car drives clockwise around the track in Figure 13.48, slowing down at the curves and speeding up along the straight portions. Sketch velocity vectors at the points P, Q, and R.

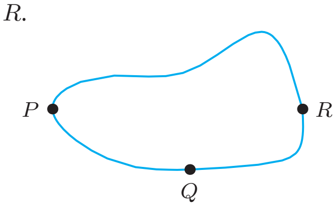

Figure 13.48

40. A racing car drives clockwise around the track shown in Figure 13.48 at a constant speed. At what point on the track does the car have the longest acceleration vector, and in roughly what direction is it pointing? (Recall that acceleration is the rate of change of velocity.)

41. List any vectors which are parallel to each other and any vectors which are perpendicular to each other:

$\vec{v}_1=\vec{i}-2\vec{j}$ $\quad \vec{v}_2=2\vec{i}+4\vec{j}$
$\vec{v}_3=3\vec{i}+1.5\vec{j}$ $\quad \vec{v}_4=-1.2\vec{i}+2.4\vec{j}$
$\vec{v}_5=-5\vec{i}-2.5\vec{j}$ $\quad \vec{v}_6=12\vec{i}-12\vec{j}$
$\vec{v}_7=4\vec{i}+2\vec{j}$ $\quad \vec{v}_8=3\vec{i}-6\vec{j}$
$\vec{v}_9=0.70\vec{i}-0.35\vec{j}$

42. An object is to be moved vertically upward by a crane. As the crane cannot get directly above the object, three ropes are attached to guide the object. One rope is pulled parallel to the ground with a force of 100 newtons in a direction 30° north of east. The second rope is pulled parallel to the ground with a force of 70 newtons in a direction 80° south of east. If the crane is attached to the third rope and can pull with a total force of 3000 newtons, find the force vector for the crane. What is the resulting (total) force on the object? (Assume vector $\vec{i}$ points east, vector $\vec{j}$ points north, and vector $\vec{k}$ points vertically up.)

43. For what values of t are $\vec{u} = t\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = t\vec{i} + t\vec{j} - 2\vec{k}$ perpendicular? Are there values of t for which $\vec{u}$ and $\vec{v}$ are parallel?

In Problems 44–45, find an equation of a plane that satisfies the given conditions.

44. Perpendicular to the vector $-\vec{i} + 2\vec{j} + \vec{k}$ and passing through the point $(1, 0, 2)$.

45. Perpendicular to the vector $5\vec{i} + \vec{j} - 2\vec{k}$ and passing through the point $(0, 1, -1)$.

46. Let $A = (0, 4)$, $B = (-1, -3)$, and $C = (-5, 1)$. Draw triangle ABC and find each of its interior angles.

47. Find the area of the triangle with vertices $P = (-2, 2, 0)$, $Q = (1, 3, -1)$, and $R = (-4, 2, 1)$.

48. A plane is drawn through the points $A = (2, 1, 0)$, $B = (0, 1, 3)$ and $C = (1, 0, 1)$. Find

(a) Two vectors lying in the plane.
(b) A vector perpendicular to the plane.
(c) The equation of the plane.

49. Given the points $P = (1, 2, 3)$, $Q = (3, 5, 7)$, and $R = (2, 5, 3)$, find:

(a) A unit vector perpendicular to a plane containing P, Q, R.
(b) The angle between PQ and PR.
(c) The area of the triangle PQR.
(d) The distance from R to the line through P and Q.

50. Find the distance from the point $P = (2, -1, 3)$ to the plane $2x + 4y - z = -1$.

51. Find an equation of the plane passing through the three points $(1, 1, 1)$, $(1, 4, 5)$, $(-3, -2, 0)$. Find the distance from the origin to the plane.

52. Suppose $\vec{v} \cdot \vec{w} = 8$ and $\vec{v} \times \vec{w} = 12\vec{i} - 3\vec{j} + 4\vec{k}$ and that the angle between $\vec{v}$ and $\vec{w}$ is θ. Find

(a) $\tan\theta$ **(b)** θ

53. Consider the plane $5x - y + 7z = 21$.

(a) Find a point on the x-axis on this plane.
(b) Find two other points on the plane.
(c) Find a vector perpendicular to the plane.
(d) Find a vector parallel to the plane.

54. An airport is at the point $(200, 10, 0)$ and an approaching plane is at the point $(550, 60, 4)$. Assume that the xy-plane is horizontal, with the x-axis pointing eastward and the y-axis pointing northward. Also assume that the z-axis is upward and that all distances are measured in kilometers. The plane flies due west at a constant altitude at a speed of 500 km/hr for half an hour. It then descends at 200 km/hr, heading straight for the airport.

(a) Find the velocity vector of the plane while it is flying at constant altitude.
(b) Find the coordinates of the point at which the plane starts to descend.
(c) Find a vector representing the velocity of the plane when it is descending.

55. Find the vector $\vec{v}$ with all of the following properties:

- Magnitude 10
- Angle of 45° with positive x-axis
- Angle of 75° with positive y-axis
- Positive $\vec{k}$ -component.

56. Two lines in space are skew if they are not parallel and do not intersect. Determine the minimum distance between two such lines.

57. (a) A vector $\vec{v}$ of magnitude v makes an angle α with the positive x-axis, angle β with the positive y-axis, and angle γ with the positive z-axis. Show that

$$\vec{v} = v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}.$$

(b) $\text{Cos}\,\alpha$, $\cos\beta$, and $\cos\gamma$ are called *direction cosines*. Show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

58. Three people are trying to hold a ferocious lion still for the veterinarian. The lion, in the center, is wearing a collar with three ropes attached to it and each person has hold of a rope. Charlie is pulling in the direction 62° west of north with a force of 175 newtons and Sam is pulling in the direction 43° east of north with a force of 200 newtons. What is the direction and magnitude of the force which must be exerted by Alice on the third rope to counterbalance Sam and Charlie?

CAS Challenge Problems

59. Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{b} = u\vec{i} + v\vec{j} + w\vec{k}$, and $\vec{c} = m\vec{a} + n\vec{b}$. Compute $(\vec{a} \times \vec{b}) \cdot \vec{c}$ and $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$, and explain the geometric meaning of your answers.

60. Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{b} = u\vec{i} + v\vec{j} + w\vec{k}$ and $\vec{c} = r\vec{i} + s\vec{j} + t\vec{k}$. Show that the parallelepiped with edges $\vec{a}$, $\vec{b}$, $\vec{c}$ has the same volume as the parallelepiped with edges $\vec{a}$, $\vec{b}$, $2\vec{a} - \vec{b} + \vec{c}$. Explain this result geometrically.

61. Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

(a) For $\vec{c} = x\vec{i} + y\vec{j} + z\vec{k}$, write the following conditions as equations in x, y, z and solve them:

$$\vec{a} \cdot \vec{c} = 0, \quad \vec{b} \cdot \vec{c} = 0, \quad \|\vec{c}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta.$$

[Hint: Use the dot product to find $\sin^2\theta$.]

(b) Compute the cross product $\vec{a} \times \vec{b}$ and compare with your answer in part (a). What do you notice? Explain.

62. Let $A = (0,0,0)$, $B = (2,0,0)$, $C = (1, \sqrt{3}, 0)$ and $D = (1, 1/\sqrt{3}, 2\sqrt{2/3})$.

(a) Show that A, B, C, D are all the same distance from each other.

(b) Find the point $P = (x, y, z)$ which is equidistant from A, B, C and D by setting up and solving three equations in x, y, and z.

(c) Use the dot product to find the angle APB. (In chemistry, this angle is often approximated by $109.5°$. A methane molecule can be represented by four hydrogen atoms at points A, B, C and D, and a carbon atom at P.)

63. Let $P = (x, y, z)$, $Q = (u, v, w)$ and $R = (r, s, t)$ be points on the plane $ax + by + cz = d$.

(a) What is the relation between $\overrightarrow{PQ} \times \overrightarrow{PR}$ and the normal vector to the plane, $a\vec{i} + b\vec{j} + c\vec{k}$?

(b) Express $\overrightarrow{PQ} \times \overrightarrow{PR}$ in terms of $x, y, z, u, v, w, r, s, t$.

(c) Use the equation for the plane to eliminate z, w, and t from the expression you obtained in part (b), and simplify. Does your answer agree with what you said in part (a)?

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–30 true or false? Give reasons for your answer.

1. There is exactly one unit vector parallel to a given nonzero vector $\vec{v}$.

2. The vector $\dfrac{1}{\sqrt{3}}\vec{i} + \dfrac{-1}{\sqrt{3}}\vec{j} + \dfrac{2}{\sqrt{3}}\vec{k}$ is a unit vector.

3. The length of the vector $2\vec{v}$ is twice the length of the vector $\vec{v}$.

4. If $\vec{v}$ and $\vec{w}$ are any two vectors, then $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.

5. If $\vec{v}$ and $\vec{w}$ are any two vectors, then $\|\vec{v} - \vec{w}\| = \|\vec{v}\| - \|\vec{w}\|$.

6. The vectors $2\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$ are parallel.

7. The vector $\vec{u} + \vec{v}$ is always larger in magnitude than both $\vec{u}$ and $\vec{v}$.

8. For any scalar c and vector $\vec{v}$ we have $\|c\vec{v}\| = c\|\vec{v}\|$.

9. The displacement vector from $(1,1,1)$ to $(1,2,3)$ is $-\vec{j} - 2\vec{k}$.

10. The displacement vector from (a, b) to (c, d) is the same as the displacement vector from (c, d) to (a, b).

11. The quantity $\vec{u} \cdot \vec{v}$ is a vector.

12. The plane $x + 2y - 3z = 5$ has normal vector $\vec{i} + 2\vec{j} - 3\vec{k}$.

13. If $\vec{u} \cdot \vec{v} < 0$ then the angle between $\vec{u}$ and $\vec{v}$ is greater than $\pi/2$.

14. An equation of the plane with normal vector $\vec{i} + \vec{j} + \vec{k}$ containing the point $(1,2,3)$ is $z = x + y$.

15. The triangle in 3-space with vertices $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$ has a right angle.

16. The dot product $\vec{v} \cdot \vec{v}$ is never negative.

17. If $\vec{u} \cdot \vec{v} = 0$ then either $\vec{u} = 0$ or $\vec{v} = 0$.

18. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all nonzero, and $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{v} = \vec{w}$.

19. For any vectors $\vec{u}$ and $\vec{v}$: $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

20. If $\|\vec{u}\| = 1$, then the vector $\vec{v} - (\vec{v} \cdot \vec{u})\vec{u}$ is perpendicular to $\vec{u}$.

21. $\vec{u} \times \vec{v}$ is a vector.

22. $\vec{u} \times \vec{v}$ has direction parallel to both $\vec{u}$ and $\vec{v}$.

23. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|$.

24. $(\vec{i} \times \vec{j}) \cdot \vec{k} = \vec{i} \cdot (\vec{j} \times \vec{k})$.

25. If $\vec{v}$ is a nonzero vector and $\vec{v} \times \vec{u} = \vec{v} \times \vec{w}$, then $\vec{u} = \vec{w}$.

26. The value of $\vec{v} \cdot (\vec{v} \times \vec{w})$ is always 0.

27. The value of $\vec{v} \times \vec{w}$ is never the same as $\vec{v} \cdot \vec{w}$.

28. The area of the triangle with two sides given by $\vec{i} + \vec{j}$ and $\vec{j} + 2\vec{k}$ is $3/2$.

29. Given a nonzero vector $\vec{v}$ in 3-space, there is a nonzero vector $\vec{w}$ such that $\vec{v} \times \vec{w} = \vec{0}$.

30. It is never true that $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

PROJECTS FOR CHAPTER THIRTEEN

1. Cross Product of Vectors in the Plane

Let $\vec{a} = a_1\vec{i} + a_2\vec{j}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j}$ be two nonparallel vectors in 2-space, as in Figure 13.49.

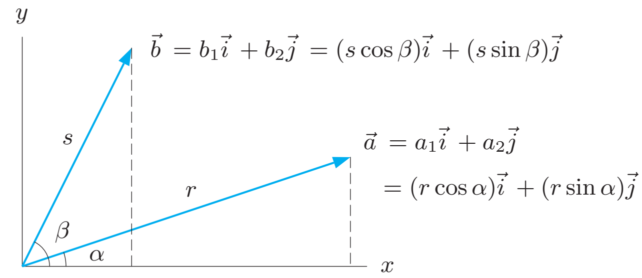

Figure 13.49

(a) Use the identity $\sin(\beta - \alpha) = (\sin\beta\cos\alpha - \cos\beta\sin\alpha)$ to derive the formula for the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$:

$$\text{Area of parallelogram} = |a_1 b_2 - a_2 b_1|.$$

(b) Show that $a_1b_2 - a_2b_1$ is positive when the rotation from $\vec{a}$ to $\vec{b}$ is counterclockwise, and negative when it is clockwise.

(c) Use parts (a) and (b) to show that the geometric and algebraic definitions of $\vec{a} \times \vec{b}$ give the same result.

2. The Dot Product in Genetics[2]

We define[3] the angle between two n-dimensional vectors, $\vec{v}$ and $\vec{w}$, using the dot product:

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{v_1w_1 + v_2w_2 + \cdots + v_nw_n}{\|\vec{v}\|\|\vec{w}\|}, \qquad \text{provided } \|\vec{v}\|, \|\vec{w}\| \neq 0.$$

We use this idea of angle to measure how close two populations are to one another genetically. The table shows the relative frequencies of four alleles (variants of a gene) in four populations.

Allele	Eskimo	Bantu	English	Korean
A_1	0.29	0.10	0.21	0.22
A_2	0.00	0.09	0.07	0.00
B	0.03	0.12	0.06	0.21
O	0.68	0.69	0.66	0.57

Let $\vec{a}_1$ be the 4-vector showing the square roots of the relative frequencies of the alleles in the Eskimo population. Let $\vec{a}_2, \vec{a}_3, \vec{a}_4$ be the corresponding vectors for the Bantu, English, and Korean populations, respectively. The genetic distance between two populations is defined as the angle between the corresponding vectors.

(a) Using this definition, is the English population closer genetically to the Bantus or to the Koreans? Explain.

(b) Is the English population closer to a half Eskimo, half Bantu population than to the Bantu population alone?

(c) Among all possible populations that are a mix of Eskimo and Bantu, find the mix that is closest to the English population.

[2]Adapted from Cavalli-Sforza and Edwards, "Models and Estimation Procedures," Am J. Hum. Genet., Vol. 19 (1967), pp. 223-57.

[3]The result of Problem 63 on page 710 shows that the quantity on the right-hand side of this equation is between -1 and 1, so this definition makes sense.

3. A Warren Truss

A Warren Truss is a structure for bearing a weight such as a roof or a bridge with two supports at either end of a gap. The truss in Figure 13.50 is loaded by weights at points D and E and is supported by vertical forces at points A and C. The horizontal bars in the truss are 10 ft long and the diagonal bars are 12 ft. Angles A and C are $65.38°$.

Each bar exerts a force at the two joints at its ends. The two force vectors are parallel to the bar, equal in magnitude, and opposite in direction. If the bar pushes on the joints at its ends, then the bar is under compression, and if it pulls it is under tension, and the magnitude of the force is called the magnitude of the tension or compression.

Engineers need to know the magnitude of the compression or tension in each of the bars of the truss to prevent them from bending or breaking. To determine these magnitudes, we use the fact that at each joint the sum of the external forces from the weights and supports and the pushing and pulling forces exerted by the bars is zero. Find the magnitudes for all seven bars in this order:

(a) Joint A; Bars AB, AE
(b) Joint C; Bars BC, CD
(c) Joint D; Bars BD, DE
(d) Joint E; Bar BE

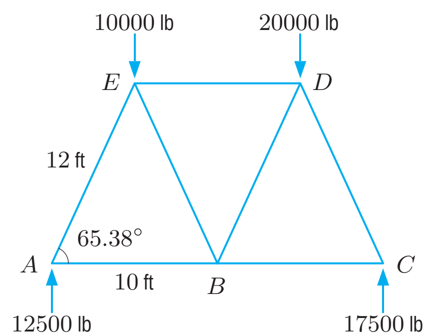

Figure 13.50