Reading content

CHAPTER

12 Functions of Several Variables

12.1 Functions of Two Variables

Function Notation

Suppose you want to calculate your monthly payment on a five-year car loan; this depends on both the amount of money you borrow and the interest rate. These quantities can vary separately: the loan amount can change while the interest rate remains the same, or the interest rate can change while the loan amount remains the same. To calculate your monthly payment you need to know both. If the monthly payment is $$m$, the loan amount is $$L$, and the interest rate is r%, then we express the fact that m is a function of L and r by writing:

$$m = f(L, r).$$

This is just like the function notation of one-variable calculus. The variable m is called the dependent variable, and the variables L and r are called the independent variables. The letter f stands for the *function* or rule that gives the value of m corresponding to given values of L and r.

A function of two variables can be represented graphically, numerically by a table of values, or algebraically by a formula. In this section, we give examples of each.

Graphical Example: A Weather Map

Figure 12.1 shows a weather map from a newspaper. What information does it convey? It displays the predicted high temperature, T, in degrees Fahrenheit (°F), throughout the US on that day. The curves on the map, called *isotherms*, separate the country into zones, according to whether T is in the 60s, 70s, 80s, 90s, or 100s. (*Iso* means same and *therm* means heat.) Notice that the isotherm separating the 80s and 90s zones connects all the points where the temperature is exactly 90°F.

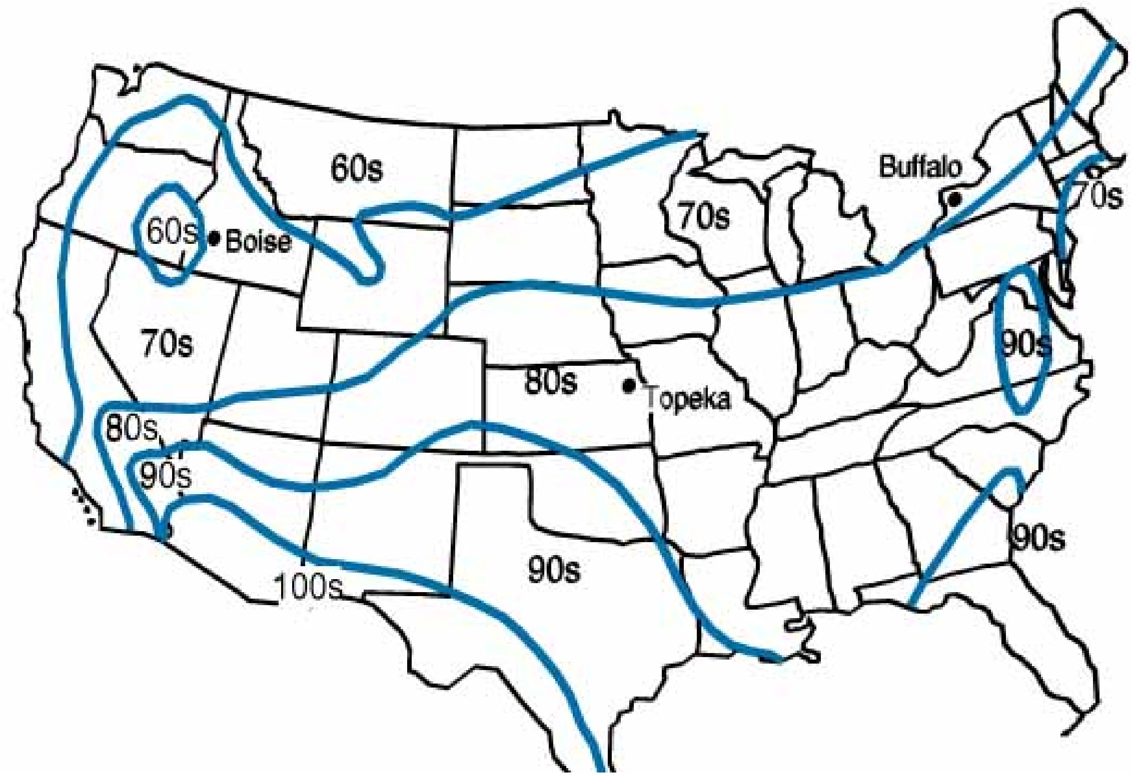

Figure 12.1 Weather map showing predicted high temperatures, T, on a summer day

Example 1

Estimate the predicted value of T in Boise, Idaho; Topeka, Kansas; and Buffalo, New York.

Solution

Boise and Buffalo are in the 70s region, and Topeka is in the 80s region. Thus, the predicted temperature in Boise and Buffalo is between 70 and 80 while the predicted temperature in Topeka is between 80 and 90. In fact, we can say more. Although both Boise and Buffalo are in the 70s, Boise is quite close to the $T = 70$ isotherm, whereas Buffalo is quite close to the $T = 80$ isotherm. So we estimate the temperature to be in the low 70s in Boise and in the high 70s in Buffalo. Topeka is about halfway between the $T = 80$ isotherm and the $T = 90$ isotherm. Thus, we guess the temperature in Topeka to be in the mid 80s. In fact, the actual high temperatures for that day were 71°F for Boise, 79°F for Buffalo, and 86°F for Topeka.

The predicted high temperature, T, illustrated by the weather map is a function of (that is, depends on) two variables, often longitude and latitude, or miles east-west and miles north-south of a fixed point, say, Topeka. The weather map in Figure 12.1 is called a *contour map* or *contour diagram* of that function. Section 12.2 shows another way of visualizing functions of two variables using surfaces; Section 12.3 looks at contour maps in detail.

Numerical Example: Beef Consumption

Suppose you are a beef producer and you want to know how much beef people will buy. This depends on how much money people have and on the price of beef. The consumption of beef, C (in pounds per week per household) is a function of household income, I (in thousands of dollars per year), and the price of beef, p (in dollars per pound). In function notation, we write:

$$C = f(I, p).$$

Table 12.1 contains values of this function. Values of p are shown across the top, values of I are down the left side, and corresponding values of $f(I, p)$ are given in the table.[1] For example, to find the value of $f(40, 3.50)$, we look in the row corresponding to $I = 40$ under $p = 3.50$, where we find the number 4.05. Thus,

$$f(40, 3.50) = 4.05$$

This means that, on average, if a household's income is $40,000 a year and the price of beef is $3.50/lb, the family will buy 4.05 lbs of beef per week.

Table 12.1 *Quantity of Beef Bought (Pounds/Household/Week)*

		Price of beef, ($/lb)			
		3.00	3.50	4.00	4.50
Household income per year, I (1000)	20	2.65	2.59	2.51	2.43
	40	4.14	4.05	3.94	3.88
	60	5.11	5.00	4.97	4.84
	80	5.35	5.29	5.19	5.07
	100	5.79	5.77	5.60	5.53

Notice how this differs from the table of values of a one-variable function, where one row or one column is enough to list the values of the function. Here many rows and columns are needed because the function has a value for every *pair* of values of the independent variables.

Algebraic Examples: Formulas

In both the weather map and beef consumption examples, there is no formula for the underlying function. That is usually the case for functions representing real-life data. On the other hand, for many idealized models in physics, engineering, or economics, there are exact formulas.

Example 2

Give a formula for the function $M = f(B, t)$ where M is the amount of money in a bank account t years after an initial investment of B dollars, if interest is accrued at a rate of 5% per year compounded annually.

Solution

Annual compounding means that M increases by a factor of 1.05 every year, so

$$M = f(B, t) = B(1.05)^t.$$

Example 3

A cylinder with closed ends has radius r and height h. If its volume is V and its surface area is A, find formulas for the functions $V = f(r, h)$ and $A = g(r, h)$.

Solution

Since the area of the circular base is πr^2, we have

$$V = f(r, h) = \text{Area of base} \cdot \text{Height} = \pi r^2 h.$$

The surface area of the side is the circumference of the bottom, $2\pi r$, times the height h, giving $2\pi rh$. Thus,

$$A = g(r, h) = 2 \cdot \text{Area of base} + \text{Area of side} = 2\pi r^2 + 2\pi rh.$$

A Tour of 3-Space

In Section 12.2 we see how to visualize a function of two variables as a surface in space. Now we see how to locate points in three-dimensional space (3-space).

Imagine three coordinate axes meeting at the *origin*: a vertical axis, and two horizontal axes at right angles to each other. (See Figure 12.2.) Think of the *xy*-plane as being horizontal, while the *z*-axis extends vertically above and below the plane. The labels *x*, *y*, and *z* show which part of each axis is positive; the other side is negative. We generally use *right-handed axes* in which looking down the positive *z*-axis gives the usual view of the *xy*-plane. We specify a point in 3-space by giving its coordinates (x, y, z) with respect to these axes. Think of the coordinates as instructions telling you how to get to the point; start at the origin, go *x* units along the *x*-axis, then *y* units in the direction parallel to the *y*-axis and finally *z* units in the direction parallel to the *z*-axis. The coordinates can be positive, zero or negative; a zero coordinate means “don't move in this direction,” and a negative coordinate means “go in the negative direction parallel to this axis.” For example, the origin has coordinates (0, 0, 0), since we get there from the origin by doing nothing at all.

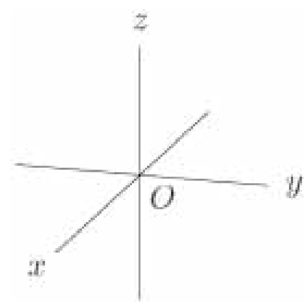

Figure 12.2 Coordinate axes in three-dimensional space

Example 4

Describe the position of the points with coordinates (1, 2, 3) and (0, 0, -1).

Solution

We get to the point (1, 2, 3) by starting at the origin, going 1 unit along the *x*-axis, 2 units in the direction parallel to the *y*-axis, and 3 units up in the direction parallel to the *z*-axis. (See Figure 12.3.)

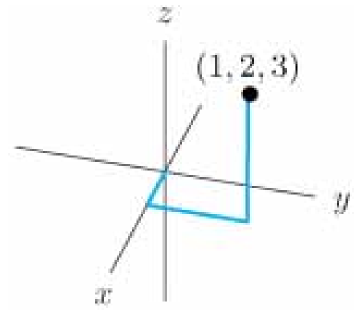

Figure 12.3 The point (1, 2, 3) in 3-space

To get to (0, 0, -1), we don't move at all in the *x* and *y* directions, but move 1 unit in the negative *z* direction. So the point is on the negative *z*-axis. (See Figure 12.4.) You can check that the position of the point is independent of the order of the *x*, *y*, and *z* displacements.

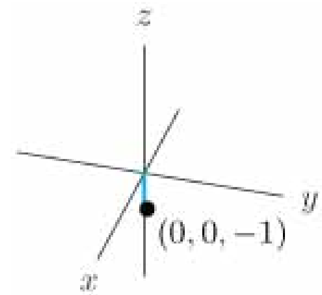

Figure 12.4 The point (0, 0, -1) in 3-space

Example 5

You start at the origin, go along the *y*-axis a distance of 2 units in the positive direction, and then move vertically upward a distance of 1 unit. What are the coordinates of your final position?

Solution

You started at the point (0, 0, 0). When you went along the *y*-axis, your *y*-coordinate increased to 2. Moving vertically increased your *z*-coordinate to 1; your *x*-coordinate did not change because you did not move in the *x* direction. So your final coordinates are (0, 2, 1). (See Figure 12.5.)

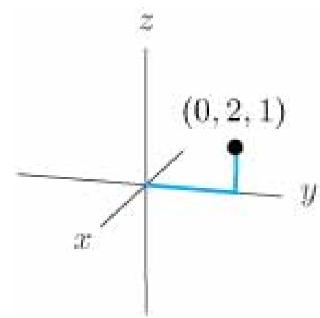

Figure 12.5 The point (0, 2, 1) is reached by moving 2 along the *y*-axis and 1 upward

It is often helpful to picture a three dimensional coordinate system in terms of a room. The origin is a corner at floor level where two walls meet the floor. The *z*-axis is the vertical intersection of the two walls; the *x*- and *y*-axes are the intersections of each wall with the floor. Points with negative coordinates lie behind a wall in the next room or below the floor.

Graphing Equations in 3-Space

We can graph an equation involving the variables *x*, *y*, and *z* in 3-space; such a graph is a picture of all points (*x*, *y*, *z*) that satisfy the equation.

Example 6

What do the graphs of the equations $z = 0$, $z = 3$, and $z = -1$ look like?

Solution

To graph $z = 0$, we visualize the set of points whose z-coordinate is zero. If the z-coordinate is 0, then we must be at the same vertical level as the origin, that is, we are in the horizontal plane containing the origin. So the graph of $z = 0$ is the middle plane in Figure 12.6. The graph of $z = 3$ is a plane parallel to the graph of $z = 0$, but three units above it. The graph of $z = -1$ is a plane parallel to the graph of $z = 0$, but one unit below it.

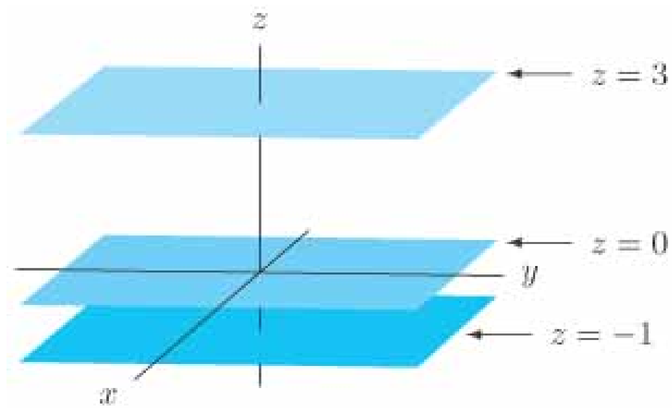

Figure 12.6 The planes $z = -1$, $z = 0$, and $z = 3$

The plane $z = 0$ contains the x- and y-coordinate axes, and is called the xy-plane. There are two other coordinate planes. The yz-plane contains both the y- and the z-axes, and the xz-plane contains the x- and z-axes. (See Figure 12.7.)

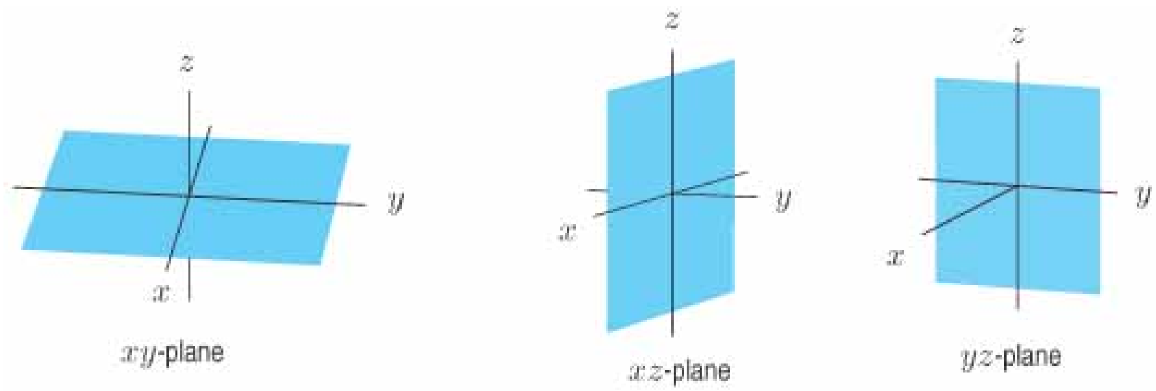

Figure 12.7 The three coordinate planes

Example 7

Which of the points $A = (1, -1, 0)$, $B = (0, 3, 4)$, $C = (2, 2, 1)$, and $D = (0, -4, 0)$ lies closest to the *xz*-plane? Which point lies on the *y*-axis?

Solution

The magnitude of the *y*-coordinate gives the distance to the *xz*-plane. The point *A* lies closest to that plane, because it has the smallest *y*-coordinate in magnitude. To get to a point on the *y*-axis, we move along the *y*-axis, but we don't move at all in the *x* or *z* directions. Thus, a point on the *y*-axis has both its *x*- and *z*-coordinates equal to zero. The only point of the four that satisfies this is *D*. (See Figure 12.8.)

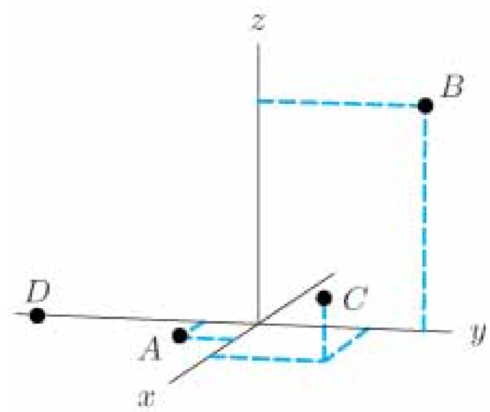

Figure 12.8 Which point lies closest to the *xz*-plane? Which point lies on the *y*-axis?

In general, if a point has one of its coordinates equal to zero, it lies in one of the coordinate planes. If a point has two of its coordinates equal to zero, it lies on one of the coordinate axes.

Example 8

You are 2 units below the *xy*-plane and in the *yz*-plane. What are your coordinates?

Solution

Since you are 2 units below the *xy*-plane, your *z*-coordinate is -2. Since you are in the *yz*-plane,your *x*-coordinate is 0; your *y*-coordinate can be anything. Thus, you are at the point (0, *y*, -2). The set of all such points forms a line parallel to the *y*-axis, 2 units below the *xy*-plane, and in the *yz*-plane. (See Figure 12.9.)

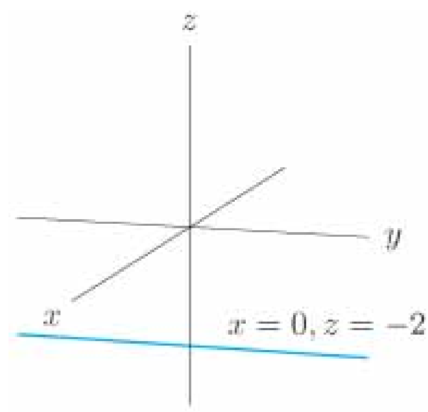

Figure 12.9 The line $x = 0$, $z = -2$

Example 9

You are standing at the point (4, 5, 2), looking at the point (0.5, 0, 3). Are you looking up or down?

Solution

The point you are standing at has z-coordinate 2, whereas the point you are looking at has z-coordinate 3; hence you are looking up.

Example 10

Imagine that the *yz*-plane in Figure 12.7 is a page of this book. Describe the region behind the page algebraically.

Solution

The positive part of the *x*-axis pokes out of the page; moving in the positive *x* direction brings you out in front of the page. The region behind the page corresponds to negative values of *x*, so it is the set of all points in 3-space satisfying the inequality $x < 0$.

Distance between Two Points

In 2-space, the formula for the distance between two points (x, y) and (a, b) is given by

$$\text{Distance} = \sqrt{(x-a)^2 + (y-b)^2}.$$

The distance between two points (x, y, z) and (a, b, c) in 3-space is represented by *PG* in Figure 12.10. The side *PE* is parallel to the *x*-axis, *EF* is parallel to the *y*-axis, and *FG* is parallel to the *z*-axis.

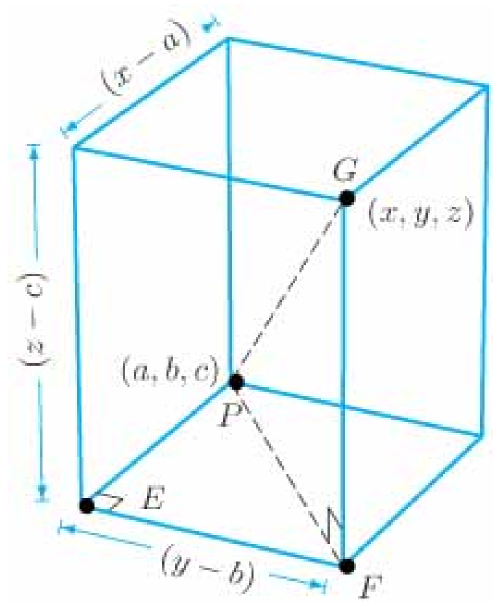

Figure 12.10 The diagonal *PG* gives the distance between the points (x, y, z) and (a, b, c)

Using Pythagoras' theorem twice gives

$$(PG)^2 = (PF)^2 + (FG)^2 = (PE)^2 + (EF)^2 + (FG)^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

Thus, a formula for the distance between the points (x, y, z) and (a, b, c) in 3-space is

$$\text{Distance} = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Example 11

Find the distance between (1, 2, 1) and (-3, 1, 2).

Solution

$$\text{Distance} = \sqrt{(-3-1)^2 + (1-2)^2 + (2-1)^2} = \sqrt{18} = 4.24.$$

Example 12

Find an expression for the distance from the origin to the point (x, y, z).

Solution

The origin has coordinates (0, 0, 0), so the distance from the origin to (x, y, z) is given by

$$\text{Distance} = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

Example 13

Find an equation for a sphere of radius 1 with center at the origin.

Solution

The sphere consists of all points (x, y, z) whose distance from the origin is 1, that is, which satisfy the equation

$$\sqrt{x^2 + y^2 + z^2} = 1.$$

This is an equation for the sphere. If we square both sides we get the equation in the form

$$x^2 + y^2 + z^2 = 1.$$

Note that this equation represents the *surface* of the sphere. The solid ball enclosed by the sphere is represented by the inequality $x^2 + y^2 + z^2 \leq 1$.

Exercises and Problems for Section 12.1

Exercises

1. Which of the points $A = (23, 92, 48)$, $B = (-60, 0, 0)$, $C = (60, 1, -92)$ is closest to the yz-plane? Which lies on the xz-plane? Which is farthest from the xy-plane?

2. Which of the points $A = (1.3, -2.7, 0)$, $B = (0.9, 0, 3.2)$, $C = (2.5, 0.1, -0.3)$ is closest to the yz-plane? Which one lies on the xz-plane? Which one is farthest from the xy-plane?

3. Which of the points $P = (1, 2, 1)$ and $Q = (2, 0, 0)$ is closest to the origin?

4. Which two of the three points $P_1 = (1, 2, 3)$, $P_2 = (3, 2, 1)$ and $P_3 = (1, 1, 0)$ are closest to each other?

5. You are at the point (3, 1, 1), standing upright and facing the yz-plane. You walk 2 units forward, turn left, and walk another 2 units. What is your final position? From the point of view of an observer looking at the coordinate system in Figure 12.2, are you in front of or behind the yz-plane? To the left or to the right of the xz-plane? Above or below the xy-plane?

6. You are at the point (-1, -3, -3), standing upright and facing the *yz*-plane. You walk 2 units forward, turn left, and walk for another 2 units. What is your final position? From the point of view of an observer looking at the coordinate system in Figure 12.2, are you in front of or behind the *yz*-plane? To the left or to the right of the *xz*-plane? Above or below the *xy*-plane?

Sketch graphs of the equations in Exercises 7, 8 and 9 in 3-space.

7. $x = -3$

8. $y = 1$

9. $z = 2$ and $y = 4$

10. Find the equation of the sphere of radius 5 centered at the origin.

11. Find the equation of the sphere of radius 5 centered at (1, 2, 3).

12. Find the equation of the vertical plane perpendicular to the *y*-axis and through the point (2, 3, 4).

Exercises 13, 14 and 15 refer to the map in Figure 12.1.

13. Give the range of daily high temperatures for:

(a) Pennsylvania

(b) North Dakota

(c) California

14. Sketch a possible graph of the predicted high temperature T on a line north-south through Topeka.

15. Sketch possible graphs of the predicted high temperature on a north-south line and an east-west line through Boise.

For Exercises 16, 17 and 18, refer to Table 12.1, where p is the price of beef and I is annual household income.

16. Give tables for beef consumption as a function of p, with I fixed at $I = 20$ and $I = 100$. Give tables for beef consumption as a function of I, with p fixed at $p = 3.00$ and $p = 4.00$. Comment on what you see in the tables.

17. Make a table of the proportion, P, of household income spent on beef per week as a function of price and income. (Note that P is the fraction of income spent on beef.)

18. How does beef consumption vary as a function of household income if the price of beef is held constant?

Problems

19. The temperature adjusted for wind-chill is a temperature which tells you how cold it feels, as a result of the combination of wind and temperature.[2] See Table 12.2.

Table 12.2 *Temperature Adjusted for Wind-Chill (°F) as a Function of Wind Speed and Temperature*

		Temperature (°F)							
		35	30	25	20	15	10	5	0
Wind Speed (mph)	5	31	25	19	13	7	1	-5	-11
	10	27	21	15	9	3	-4	-10	-16
	15	25	19	13	6	0	-7	-13	-19
	20	24	17	11	4	-2	-9	-15	-22
	25	23	16	9	3	-4	-11	-17	-24

(a) If the temperature is 0°F and the wind speed is 15 mph, how cold does it feel?

(b) If the temperature is 35°F, what wind speed makes it feel like 24°F?

(c) If the temperature is 25°F, what wind speed makes it feel like 12°F?

(d) If the wind is blowing at 20 mph, what temperature feels like 0°F?

20. Using Table 12.2, make tables of the temperature adjusted for wind-chill as a function of wind speed for temperatures of 20°F and 0°F.

21. Using Table 12.2, make tables of the temperature adjusted for wind-chill as a function of temperature for wind speeds of 5 mph and 20 mph.

22. The balance, B, in dollars, in a bank account depends on the amount deposited, A dollars, the annual interest rate, r%, and the time, t, in months since the deposit, so $B = f(A, r, t)$.

(a) Is f an increasing or decreasing function of A? Of r? Of t?

(b) Interpret the statement $f(1250, 1, 25) \approx 1276$. Give units.

23. The monthly payments, P dollars, on a mortgage in which A dollars were borrowed at an annual interest rate of r% for t years is given by $P = f(A, r, t)$. Is f an increasing or decreasing function of A? Of r? Of t?

24. A car rental company charges $40 a day and 15 cents a mile for its cars.

(a) Write a formula for the cost, C, of renting a car as a function, f, of the number of days, d, and the number of miles driven, m.

(b) If $C = f(d, m)$, find $f(5, 300)$ and interpret it.

25. The gravitational force, F newtons, exerted on an object by the earth depends on its mass, m kilograms, and its distance, r meters, from the center of the earth, so $F = f(m, r)$. Interpret the following statement in terms of gravitation: $f(100, 7000000) \approx 820$.

26. Consider the acceleration due to gravity, g, at a height h above the surface of a planet of mass m.

(a) If m is held constant, is g an increasing or decreasing function of h? Why?

(b) If h is held constant, is g an increasing or decreasing function of m? Why?

27. A cube is located such that its top four corners have the coordinates (-1, -2, 2), (-1, 3, 2), (4, -2, 2) and (4, 3, 2). Give the coordinates of the center of the cube.

28. Describe the set of points whose distance from the x-axis is 2.

29. Describe the set of points whose distance from the x-axis equals the distance from the yz-plane.

30. Find a formula for the shortest distance between a point (a, b, c) and the y-axis.

31. Find the equations of planes that just touch the sphere $(x - 2)^2 + (y - 3)^2 + (z - 3)^2 = 16$ and are parallel to

(a) The xy-plane

(b) The yz-plane

(c) The xz-plane

32. Find an equation of the largest sphere contained in the cube determined by the planes $x = 2$, $x = 6$; $y = 5$, $y = 9$; and $z = -1$, $z = 3$.

33. Which of the points $P_1 = (-3, 2, 15)$, $P_2 = (0, -10, 0)$, $P_3 = (-6, 5, 3)$ and $P_4 = (-4, 2, 7)$ is closest to $P = (6, 0, 4)$?

34. (a) Find the equations of the circles (if any) where the sphere $(x - 1)^2 + (y + 3)^2 + (z - 2)^2 = 4$ intersects each coordinate plane.

(b) Find the points (if any) where this sphere intersects each coordinate axis.

35. A rectangular solid lies with its length parallel to the *y*-axis, and its top and bottom faces parallel to the plane $z = 0$. If the center of the object is at (1, 1, -2) and it has a length of 13, a height of 5 and a width of 6, give the coordinates of all eight corners and draw the figure labeling the eight corners.