Reading content
12.1 Functions of Two Variables
12.2 Graphs of Functions of Two Variables
12.3 Contour Diagrams
12.4 Linear Functions
12.5 Functions of Three Variables
12.6 Limits and Continuity
Chapter Summary
Review Exercises and Problems for Chapter Twelve
Check Your Understanding
Projects for Chapter Twelve
Student Solutions Manual
Graphing Calculator Manual
Focus on Theory
Web Quizzes
References

12.4 Linear Functions

What is a Linear Function of Two Variables?

Linear functions played a central role in one-variable calculus because many one-variable functions have graphs that look like a line when we zoom in. In two-variable calculus, a *linear function* is one whose graph is a plane. In Chapter 14, we see that many two-variable functions have graphs which look like planes when we zoom in.

What Makes a Plane Flat?

What makes the graph of the function $z = f(x, y)$ a plane? Linear functions of *one* variable have straight line graphs because they have constant slope. On a plane, the situation is a bit more complicated. If we walk around on a tilted plane, the slope is not always the same: it depends on the direction in which we walk. However, at every point on the plane, the slope is the same as long as we choose the same direction. If we walk parallel to the x-axis, we always find ourselves walking up or down with the same slope; the same is true if we walk parallel to the y-axis. In other words, the slope ratios $\Delta z/\Delta x$ (with y fixed) and $\Delta z/\Delta y$ (with x fixed) are each constant.

Example 1

A plane cuts the z-axis at $z = 5$, has slope 2 in the x direction and slope -1 in the y direction. What is the equation of the plane?

Solution

Finding the equation of the plane means constructing a formula for the z-coordinate of the point on the plane directly above the point (x, y) in the xy-plane. To get to that point start from the point above the origin, where $z = 5$. Then walk x units in the x direction. Since the slope in the x direction is 2, the height increases by $2x$. Then walk y units in the y direction; since the slope in the y direction is -1, the height decreases by y units.

Since the height has changed by $2x - y$ units, the z-coordinate is $5 + 2x - y$. Thus, the equation for the plane is

$$z = 5 + 2x - y.$$

For any linear function, if we know its value at a point (x_0, y_0), its slope in the x direction, and its slope in the y direction, then we can write the equation of the function. This is just like the equation of a line in the one-variable case, except that there are two slopes instead of one.

If a **plane** has slope m in the x direction, slope n in the y direction, and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

This plane is the graph of the **linear function**

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0).$$

If we write $c = z_0 - mx_0 - ny_0$, then we can write $f(x, y)$ in the equivalent form

$$f(x, y) = c + mx + ny.$$

Just as in 2-space a line is determined by two points, so in 3-space a plane is determined by three points, provided they do not lie on a line.

Example 2

Find the equation of the plane passing through the points (1, 0, 1), (1, -1, 3), and (3, 0, -1).

Solution

The first two points have the same x-coordinate, so we use them to find the slope of the plane in the y-direction. As the y-coordinate changes from 0 to -1, the z-coordinate changes from 1 to 3, so the slope in the y-direction is $n = \Delta z/\Delta y = (3 - 1)/(-1 - 0) = -2$. The first and third points have the same y-coordinate, so we use them to find the slope in the x-direction; it is $m = \Delta z/\Delta x = (-1 - 1)/(3 - 1) = -1$. Because the plane passes through (1, 0, 1), its equation is

$$z = 1 - (x - 1) - 2(y - 0) \quad \text{or} \quad z = 2 - x - 2y.$$

You should check that this equation is also satisfied by the points (1, -1, 3) and (3, 0, -1).

Example 2 was made easier by the fact that two of the points had the same x-coordinate and two had the same y-coordinate. An alternative method, which works for any three points, is to substitute the x, y, and z-values of each of the three points into the equation $z = c + mx + ny$. The resulting three equations in c, m, n are then solved simultaneously.

Linear Functions from a Numerical Point of View

To avoid flying planes with empty seats, airlines sell some tickets at full price and some at a discount. Table 12.10 shows an airline's revenue in dollars from tickets sold on a particular route, as a function of the number of full-price tickets sold, f , and the number of discount tickets sold, d.

Table 12.10 *Revenue from Ticket Sales (Dollars)*

		Full-price tickets (f)			
		100	200	300	400
Distance tickets (d)	200	39, 700	63, 600	87, 500	111, 400
	400	55, 500	79, 400	103, 300	127, 200
	600	71, 300	95, 200	119, 100	143, 000
	800	87, 100	111, 000	134, 900	158, 800
	1000	102, 900	126, 800	150, 700	174, 600

In every column, the revenue jumps by $15, 800 for each extra 200 discount tickets. Thus, each column is a linear function of the number of discount tickets sold. In addition, every column has the same slope, 15,800/200 = 79 dollars/ticket. This is the price of a discount ticket. Similarly, each row is a linear function and all the rows have the same slope, 239, which is the price in dollars of a full-fare ticket. Thus, R is a linear function of f and d, given by:

$$R = 239f + 79d.$$

We have the following general result:

A **linear function** can be recognized from its table by the following features:

- Each row and each column is linear.
- All the rows have the same slope.
- All the columns have the same slope (although the slope of the rows and the slope of the columns are generally different).

Example 3

The table contains values of a linear function. Fill in the blank and give a formula for the function.

x \ y	1.5	2.0
2	0.5	1.5
3	-0.5	?

Solution

In the first column the function decreases by 1 (from 0.5 to -0.5) as x goes from 2 to 3. Since the function is linear, it must decrease by the same amount in the second column. So the missing entry must be 1.5 - 1 = 0.5. The slope of the function in the x-direction is -1. The slope in the y-direction is 2, since in each row the function increases by 1 when y increases by 0.5. From the table we get $f(2, 1.5) = 0.5$. Therefore, the formula is

$$f(x, y) = 0.5 - (x - 2) + 2(y - 1.5) = -0.5 - x + 2y.$$

What Does the Contour Diagram of a Linear Function Look Like?

The formula for the airline revenue function in Table 12.10 is $R = 239f + 79d$, where f is the number of full-fares and d is the number of discount fares sold.

Notice that the contours of this function in Figure 12.62 are parallel straight lines. What is the practical significance of the slope of these contour lines? Consider the contour $R = 100{,}000$; that means we are looking at combinations of ticket sales that yield $100,000 in revenue. If we move down and to the right on the contour, the f-coordinate increases and the d-coordinate decreases, so we sell more full-fares and fewer discount fares. This is because to receive a fixed revenue of $100,000, we must sell more full-fares if we sell fewer discount fares. The exact trade-off depends on the slope of the contour; the diagram shows that each contour has a slope of about -3. This means that for a fixed revenue, we must sell three discount fares to replace one full-fare. This can also be seen by comparing prices. Each full fare brings in $239; to earn the same amount in discount fares we need to sell $239/79 \approx 3.03 \approx 3$ fares. Since the price ratio is independent of how many of each type of fare we sell, this slope remains constant over the whole contour map; thus, the contours are all parallel straight lines.

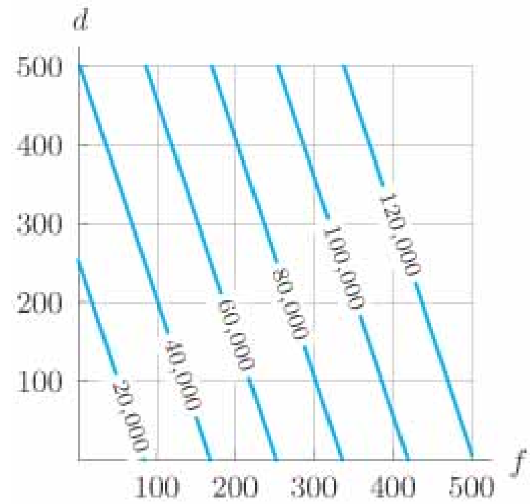


Figure 12.62 Revenue as a function of full and discount fares, $R = 239f + 79d$

Notice also that the contours are evenly spaced. Thus, no matter which contour we are on, a fixed increase in one of the variables causes the same increase in the value of the function. In terms of revenue, no matter how many fares we have sold, an extra fare, whether full or discount, brings the same revenue as before.

Example 4

Find the equation of the linear function whose contour diagram is in Figure 12.63.

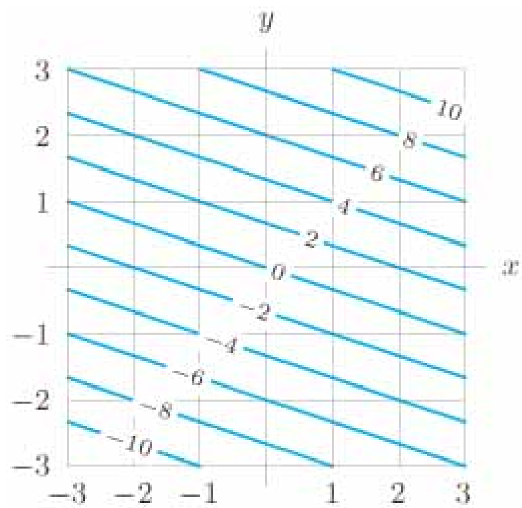


Figure 12.63 Contour map of linear function $f(x, y)$

Solution

Suppose we start at the origin on the $z = 0$ contour. Moving 2 units in the y direction takes us to the $z = 6$ contour; so the slope in the y direction is $\Delta z/\Delta y = 6/2 = 3$. Similarly, a move of

2 in the x-direction from the origin takes us to the $z = 2$ contour, so the slope in the x direction is $\Delta z/\Delta x = 2/2 = 1$. Since $f(0, 0) = 0$, we have $f(x, y) = x + 3y$.

Exercises and Problems for Section 12.4

Exercises

Problems 1 and 2 each contain a partial table of values for a linear function. Fill in the blanks.

1.

$x\backslash y$	0.0	1.0
0.0		1.0
2.0	3.0	5.0

2.

$x\backslash y$	-1.0	0.0	1.0
2.0	4.0		
3.0		3.0	5.0

Which of the tables of values in Exercises 3, 4, 5 and 6 could represent linear functions?

3.

		y		
		0	1	2
	0	0	1	4
x	1	1	0	1
	2	4	1	0

4.

		y		
		0	1	2
	0	10	13	16
x	1	6	9	12
	2	2	5	8

5.

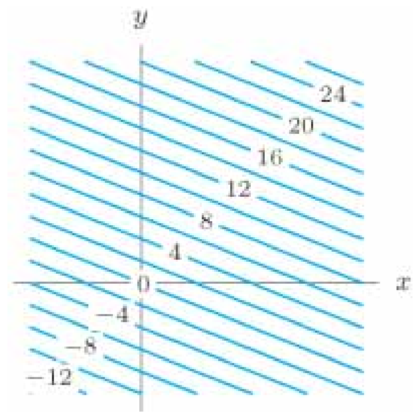
		y		
		0	1	2
	0	0	5	10
x	1	2	7	12
	2	4	9	14

6.

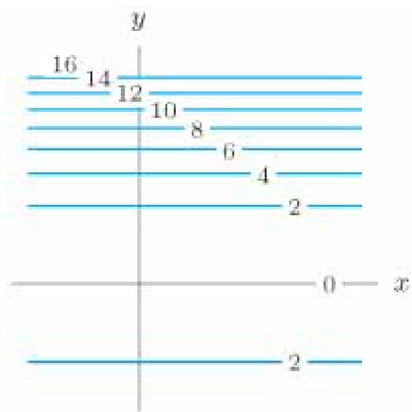
		y		
		0	1	2
	0	5	7	9
x	1	6	9	12
	2	7	11	15

Which of the contour diagrams in Exercises 7 and 8 could represent linear functions?

7.



8.



9. Find the equation of the linear function $z = c + mx + ny$ whose graph contains the points (0, 0, 0), (0, 2, -1), and (-3, 0, -4).

10. Find the linear function whose graph is the plane through the points (4, 0, 0), (0, 3, 0) and (0, 0, 2).

11. Find an equation for the plane containing the line in the xy-plane where $y = 1$, and the line in the xz-plane where $z = 2$.

12. Find the equation of the linear function $z = c + mx + ny$ whose graph intersects the xz-plane in the line $z = 3x + 4$ and intersects the yz-plane in the line $z = y + 4$.

13. Suppose that z is a linear function of x and y with slope 2 in the x direction and slope 3 in the y direction.

(a) A change of 0.5 in x and -0.2 in y produces what change in z?

(b) If $z = 2$ when $x = 5$ and $y = 7$, what is the value of z when $x = 4.9$ and $y = 7.2$?

14. (a) Find a formula for the linear function whose graph is a plane passing through point (4, 3, -2) with slope 5 in the x-direction and slope -3 in the y-direction.

(b) Sketch the contour diagram for this function.

Problems

15. A store sells CDs at one price and DVDs at another price. Figure 12.64 shows the revenue (in dollars) of the music store as a function of the number, c, of CDs and the number, d, of DVDs that it sells. What is the price of a CD? What is the price of a DVD?

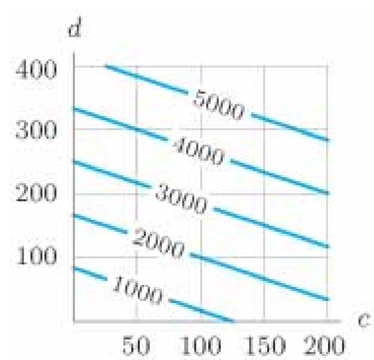


Figure 12.64

16. A college admissions office uses the following linear equation to predict the grade point average of an incoming student:

$$z = 0.003x + 0.8y - 4,$$

where z is the predicted college GPA on a scale of 0 to 4.3, and x is the sum of the student's SAT Math and SAT Verbal on a scale of 400 to 1600, and y is the student's high school GPA on a scale of 0 to 4.3. The college admits students whose predicted GPA is at least 2.3.

(a) Will a student with SATs of 1050 and high school GPA of 3.0 be admitted?

(b) Will every student with SATs of 1600 be admitted?

(c) Will every student with a high school GPA of 4.3 be admitted?

(d) Draw a contour diagram for the predicted GPA z with $400 \le x \le 1600$ and $0 \le y \le 4.3$. Shade the points corresponding to students who will be admitted.

(e) Which is more important, an extra 100 points on the SAT or an extra 0.5 of high school GPA?

17. A manufacturer makes two products out of two raw materials. Let q_1, q_2 be the quantities sold of the two products, p_1, p_2 their prices, and m_1, m_2 the quantities purchased of the two raw materials. Which of the following functions do you expect to be linear, and why? In each case, assume that all variables except the ones mentioned are held fixed.

(a) Expenditure on raw materials as a function of m_1 and m_2.

(b) Revenue as a function of q_1 and q_2.

(c) Revenue as a function of p_1 and q_1.

Problems 18, 19 and 20 concern Table 12.11, which gives the number of calories burned per minute for someone roller-blading, as a function of the person's weight and speed.[5]

Table 12.11

Calories burned per minute				
Weight	8 mph	9 mph	10 mph	11 mph
120 lbs	4.2	5.8	7.4	8.9
140 lbs	5.1	6.7	8.3	9.9
160 lbs	6.1	7.7	9.2	10.8
180 lbs	7.0	8.6	10.2	11.7
200 lbs	7.9	9.5	11.1	12.6

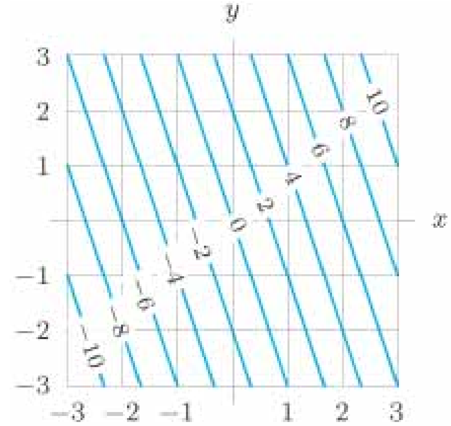
18. Does the data in Table 12.11 look approximately linear? Give a formula for B, the number of calories burned per minute in terms of the weight, w, and the speed, s. Does the formula make sense for all weights or speeds?

19. Who burns more total calories to go 10 miles: A 120 lb person going 10 mph or a 180 lb person going 8 mph? Which of these two people burns more calories per pound for the 10-mile trip?

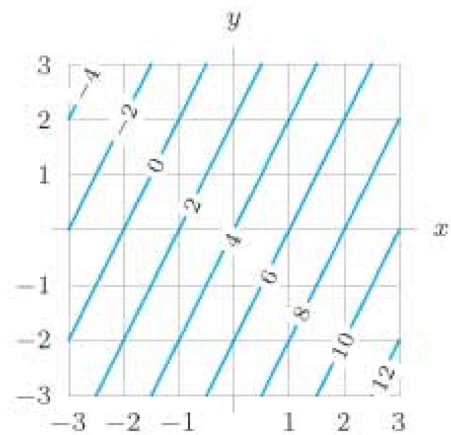
20. Use Problem 18 to give a formula for P, the number of calories burned per pound, in terms of w and s, for a person weighing w lbs roller-blading 10 miles at s mph.

For Problems 21 and 22, find possible equations for linear functions with the given contour diagrams.

21.



22.



For Problems 23 and 24, find equations for linear functions with the given values.

23.

*x**y*	-1	0	1	2
0	1.5	1	0.5	0
1	3.5	3	2.5	2
2	5.5	5	4.5	4
3	7.5	7	6.5	6

24.

*x**y*	10	20	30	40
100	3	6	9	12
200	2	5	8	11
300	1	4	7	10
400	0	3	6	9

It is difficult to graph a linear function by hand. One method that works if the x, y, and z-intercepts are positive is to plot the intercepts and join them by a triangle as shown in Figure 12.65; this shows the part of the plane in the octant where $x \geq 0$, $y \geq 0$, $z \geq 0$. If the intercepts are not all positive, the same method works if the x, y, and z-axes are drawn from a different perspective. Use this method to graph the linear functions in Problems 25, 26, 27 and 28.

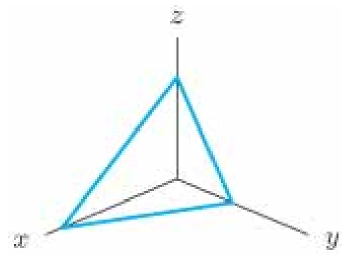


Figure 12.65

25. $z = 2 - 2x + y$

26. $z = 2 - x - 2y$

27. $z = 4 + x - 2y$

28. $z = 6 - 2x - 3y$

29. Let f be the linear function $f(x, y) = c + mx + ny$, where c, m, n are constants and $n \neq 0$.

(a) Show that all the contours of f are lines of slope $-m/n$.

(b) For all x and y, show $f(x + n, y - m) = f(x, y)$.

(c) Explain the relation between parts (a) and (b).

Problems 30 and 31 refer to the linear function $z = f(x, y)$ whose values are in Table 12.12.

Table 12.12

				y		
		4	6	8	10	12
	5	3	6	9	12	15
	10	7	10	13	16	19
x	15	11	14	17	20	23
	20	15	18	21	24	27
	25	19	22	25	28	31

30. Each column of Table 12.12 is linear with the same slope, $m = \Delta z/\Delta x = 4/5$. Each row is linear with the same slope, $n = \Delta z/\Delta y = 3/2$. We now investigate the slope obtained by moving through the table along lines that are neither rows nor columns.

(a) Move down the diagonal of the table from the upper left corner ($z = 3$) to the lower right corner ($z = 31$). What do you notice about the changes in z? Now move diagonally from $z = 6$ to $z = 27$. What do you notice about the changes in z now?

(b) Move in the table along a line right one step, up two steps from $z = 19$ to $z = 9$. Then move in the same direction from $z = 22$ to $z = 12$. What do you notice about the changes in z?

(c) Show that $\Delta z = m\Delta x + n\Delta y$. Use this to explain what you observed in parts (a) and (b).

31. If we hold y fixed, that is we keep $\Delta y = 0$, and step in the positive x-direction, we get the x-slope, m. If instead we keep $\Delta x = 0$ and step in the positive y-direction, we get the y-slope, n. Fix a step in which neither $\Delta x = 0$ nor $\Delta y = 0$. The slope in the Δx, Δy direction is

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta z}{\text{Length of step}} = \frac{\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}.$$

(a) Compute the slopes for the linear function in Table 12.12 in the direction of $\Delta x = 5$, $\Delta y = 2$.

(b) Compute the slopes for the linear function in Table 12.12 in the direction of $\Delta x = -10$, $\Delta y = 2$.