12.5 Functions of Three Variables

In applications of calculus, functions of any number of variables can arise. The density of matter in the universe is a function of three variables, since it takes three numbers to specify a point in space. Models of the US economy often use functions of ten or more variables. We need to be able to apply calculus to functions of arbitrarily many variables.

One difficulty with functions of more than two variables is that it is hard to visualize them. The graph of a function of one variable is a curve in 2-space, the graph of a function of two variables is a surface in 3-space, so the graph of a function of three variables would be a solid in 4-space. Since we can't easily visualize 4-space, we won't use the graphs of functions of three variables. On the other hand, it is possible to draw contour diagrams for functions of three variables, only now the contours are surfaces in 3-space.

Representing a Function of Three Variables using a Family of Level Surfaces

A function of two variables, $f(x, y)$, can be represented by a family of level curves of the form $f(x, y) = c$ for various values of the constant, c.

> A **level surface**, or **level set** of a function of three variables, $f(x, y, z)$, is a surface of the form $f(x, y, z) = c$, where c is a constant. The function f can be represented by the family of level surfaces obtained by allowing c to vary.

The value of the function, f, is constant on each level surface.

Example 1

The temperature, in °C, at a point (x, y, z) is given by $T = f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces of the function f look like and what do they mean in terms of temperature?

Solution

The level surface corresponding to $T = 100$ is the set of all points where the temperature is 100°C. That is, where $f(x, y, z) = 100$, so

$$x^2 + y^2 + z^2 = 100.$$

This is the equation of a sphere of radius 10, with center at the origin. Similarly, the level surface corresponding to $T = 200$ is the sphere with radius $\sqrt{200}$. The other level surfaces are concentric spheres. The temperature is constant on each sphere. We may view the temperature distribution as a set of nested spheres, like concentric layers of an onion, each one labeled with a different temperature, starting from low temperatures in the middle and getting hotter as we go out from the center. (See Figure 12.66.) The level surfaces become more closely spaced as we move farther from the origin because the temperature increases more rapidly the farther we get from the origin.

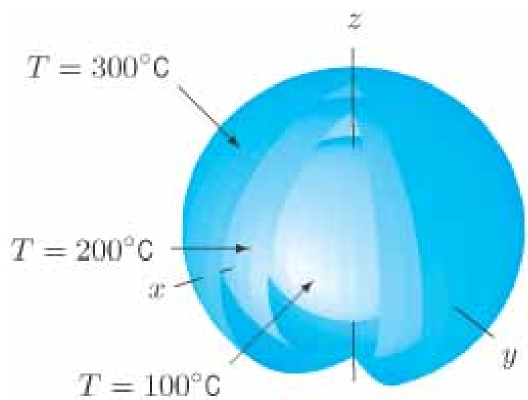

Figure 12.66 Level surfaces of $T = f(x, y, z) = x^2 + y^2 + z^2$, each one having a constant temperature

Example 2

What do the level surfaces of $f(x, y, z) = x^2 + y^2$ and $g(x, y, z) = z - y$ look like?

Solution

The level surface of f corresponding to the constant c is the surface consisting of all points satisfying the equation

$$x^2 + y^2 = c.$$

Since there is no z-coordinate in the equation, z can take any value. For $c > 0$, this is a circular cylinder of radius $\sqrt{c}$ around the z-axis. The level surfaces are concentric cylinders; on the narrow ones near the z-axis, f has small values; on the wider ones, f has larger values. See Figure 12.67.

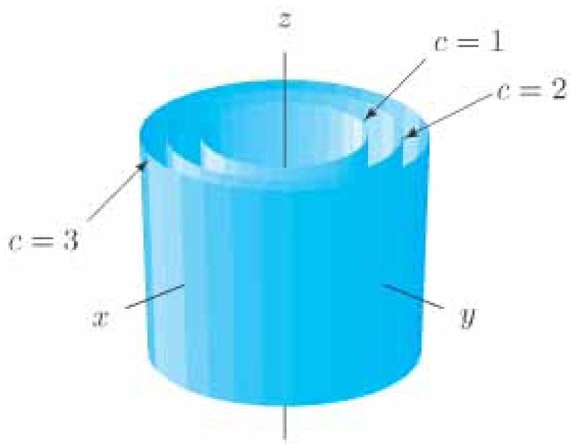

Figure 12.67 Level surfaces of $f(x, y, z) = x^2 + y^2$

The level surface of g corresponding to the constant c is the surface

$$z - y = c.$$

This time there is no x variable, so this surface is the one we get by taking each point on the straight line $z - y = c$ in the yz-plane and letting x vary. We get a plane which cuts the yz-plane diagonally; the x-axis is parallel to this plane. See Figure 12.68.

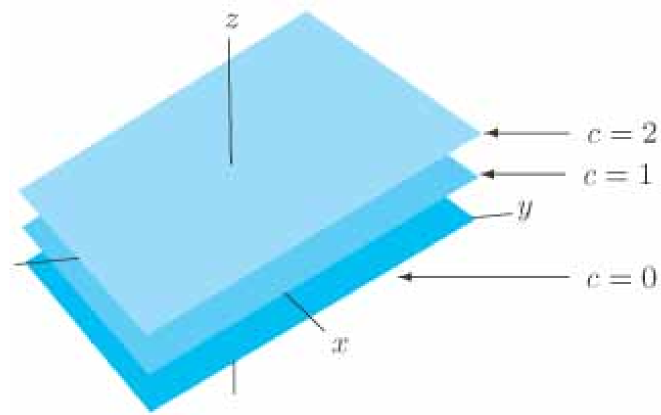

Figure 12.68 Level surfaces of $g(x, y, z) = z - y$

Example 3

What do the level surfaces of $f(x, y, z) = x^2 + y^2 - z^2$ look like?

Solution

In Section 12.3, we saw that the two-variable quadratic function $g(x, y) = x^2 - y^2$ has a saddle-shaped graph and three types of contours. The contour equation $x^2 - y^2 = c$ gives a hyperbola opening right-left when $c > 0$, a hyperbola opening up-down when $c < 0$, and a pair of intersecting lines when $c = 0$. Similarly, the three-variable quadratic function $f(x, y, z) = x^2 + y^2 - z^2$ has three types of level surfaces depending on the value of c in the equation $x^2 + y^2 - z^2 = c$.

Suppose that $c > 0$, say $c = 1$. Rewrite the equation as $x^2 + y^2 = z^2 + 1$ and think of what happens as we cut the surface perpendicular to the z-axis by holding z fixed. The result is a circle, $x^2 + y^2 =$ constant, of radius at least 1 (since the constant $z^2 + 1 \geq 1$). The circles get larger as z gets larger. If we take the $x = 0$ cross-section instead, we get the hyperbola $y^2 - z^2 = 1$. The result is shown in Figure 12.72, with $a = b = c = 1$.

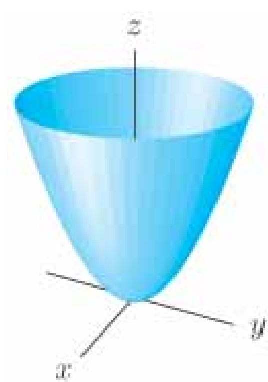

Figure 12.69 Elliptical paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

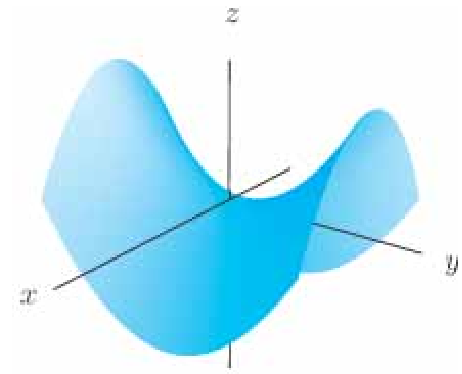

Figure 12.70 Hyperbolic paraboloid

$$z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

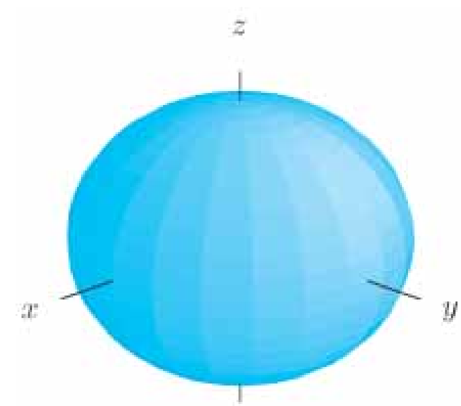

Figure 12.71 Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

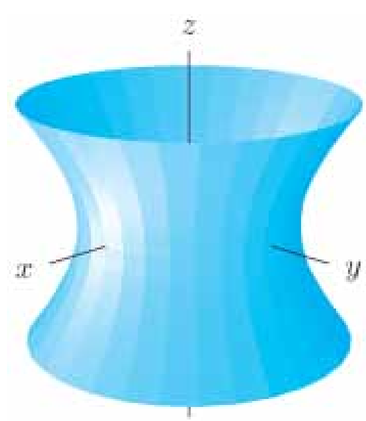

Figure 12.72 Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Suppose instead $c < 0$, say $c = -1$. Then the horizontal cross-sections of $x^2 + y^2 = z^2 - 1$ are again circles except that the radii shrink to 0 at $z = \pm 1$ and between $z = -1$ and $z = 1$ there are no cross-sections at all. The result is shown in Figure 12.73 with $a = b = c = 1$.

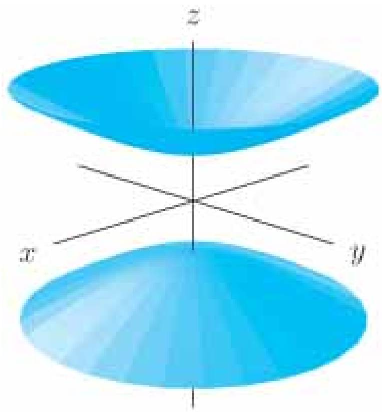

Figure 12.73 Hyperboloid of two sheets $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

When $c = 0$, we get the equation $x^2 + y^2 = z^2$. Again the horizontal cross-sections are circles, this time with the radius shrinking down to exactly 0 when $z = 0$. The resulting surface, shown in Figure 12.74 with $a = b = c = 1$, is the cone $z=\sqrt{x^2+y^2}$ studied in Section 12.3, together with the lower cone $z=-\sqrt{x^2+y^2}$.

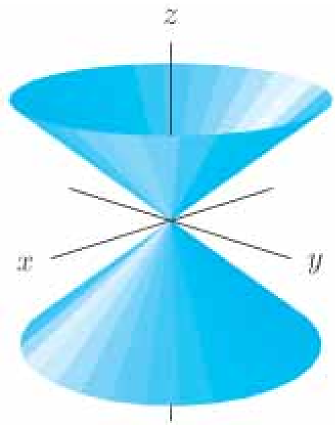

Figure 12.74 Cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

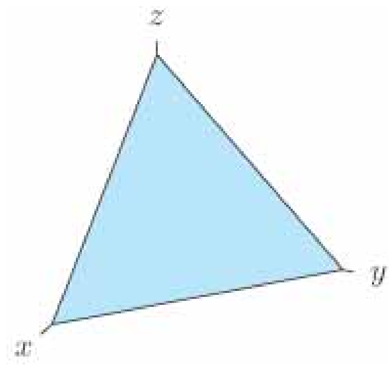

Figure 12.75 Plane $ax + by + cz = d$

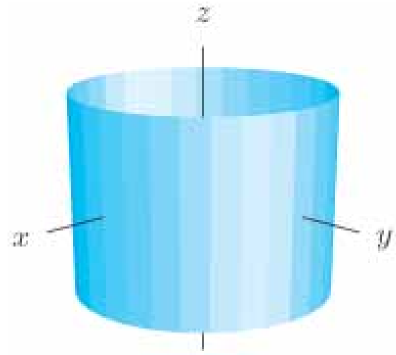

Figure 12.76 Cylindrical surface $x^2 + y^2 = a^2$

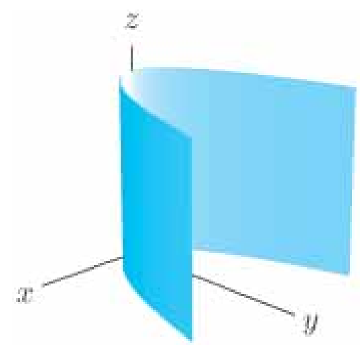

Figure 12.77 Parabolic cylinder $y = ax^2$

A Catalog of Surfaces

For later reference, here is a small catalog of the surfaces we have encountered.

(These are viewed as equations in three variables x, y, and z)

How Surfaces Can Represent Functions of Two Variables and Functions of Three Variables

You may have noticed that we have used surfaces to represent functions in two different ways. First, we used a *single* surface to represent a two-variable function $f(x, y)$. Second, we used a *family* of level surfaces to represent a three-variable function $g(x, y, z)$. These level surfaces have equation $g(x, y, z) = c$.

What is the relation between these two uses of surfaces? For example, consider the function

$$f(x, y) = x^2 + y^2 + 3.$$

Define

$$g(x, y, z) = x^2 + y^2 + 3 - z$$

The points on the graph of *f* satisfy $z = x^2 + y^2 + 3$, so they also satisfy $x^2 + y^2 + 3 - z = 0$. Thus the graph of *f* is the same as the level surface

$$g(x, y, z) = x^2 + y^2 + 3 - z = 0.$$

In general, we have the following result:

A single surface that is the graph of a two-variable function $f(x, y)$ can be thought of as one member of the family of level surfaces representing the three-variable function

$$g(x, y, z) = f(x, y) - z.$$

The graph of *f* is the level surface $g = 0$.

Conversely, a single level surface $g(x, y, z) = c$ can be regarded as the graph of a function $f(x, y)$ if it is possible to solve for *z*. Sometimes the level surface is pieced together from the graphs of two or more two-variable functions. For example, if $g(x, y, z) = x^2 + y^2 + z^2$, then one member of the family of level surfaces is the sphere

$$x^2 + y^2 + z^2 = 1.$$

This equation defines *z* implicitly as a function of *x* and *y*. Solving it gives two functions

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}.$$

The graph of the first function is the top half of the sphere and the graph of the second function is the bottom half.

Exercises and Problems for Section 12.5

Exercises

1. Match the following functions with the level surfaces in Figure 12.78.

(a) $f(x, y, z) = y^2 + z^2$

(b) $h(x, y, z) = x^2 + z^2$.

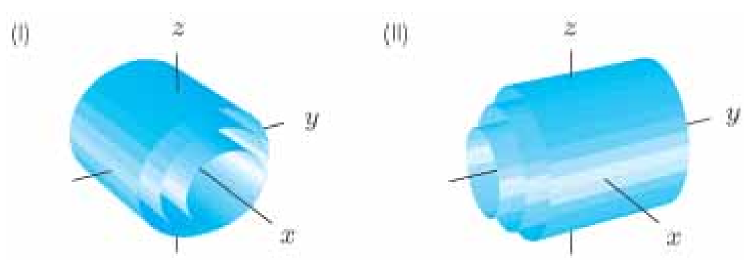

Figure 12.78

2. Match the functions with the level surfaces in Figure 12.79.

(a) $f(x, y, z) = x^2 + y^2 + z^2$

(b) $g(x, y, z) = x^2 + z^2$.

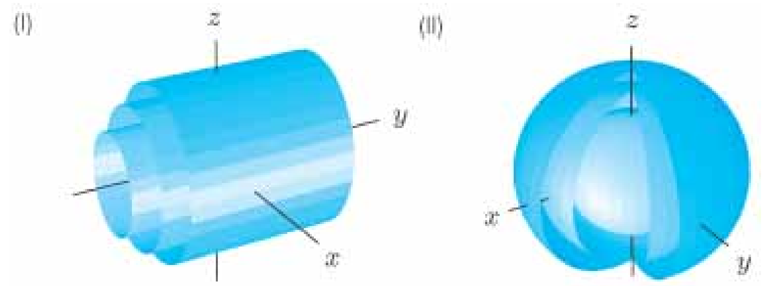

Figure 12.79

3. Write the level surface $x + 2y + 3z = 5$ as the graph of a function $f(x, y)$.

4. Find a formula for a function $f(x, y, z)$ whose level surface $f = 4$ is a sphere of radius 2, centered at the origin.

5. Write the level surface $x^2 + y + \sqrt{z} = 1$ as the graph of a function $f(x, y)$.

6. Find a formula for a function $f(x, y, z)$ whose level surfaces are spheres centered at the point (a, b, c).

7. Which of the graphs in the catalog of surfaces is the graph of a function of x and y?

Use the catalog to identify the surfaces in Exercises 8, 9, 10 and 11.

8. $x^2 + y^2 - z = 0$

9. $-x^2 - y^2 + z^2 = 1$

10. $x + y = 1$

11. $x^2 + y^2/4 + z^2 = 1$

In Exercises 12, 13, 14 and 15, decide if the given level surface can be expressed as the graph of a function, $f(x, y)$.

12. $z - x^2 - 3y^2 = 0$

13. $2x + 3y - 5z - 10 = 0$

14. $x^2 + y^2 + z^2 - 1 = 0$

15. $z^2 = x^2 + 3y^2$

Problems

In Exercises 16, 17 and 18, represent the surface whose equation is given as the graph of a two-variable function, $f(x, y)$, and as the level surface of a three-variable function, $g(x, y, z) = c$. There are many possible answers.

16. The plane $4x - y - 2z = 6$

17. The top half of the sphere $x^2 + y^2 + z^2 - 10 = 0$

18. The bottom half of the ellipsoid $x^2 + y^2 + z^2/2 = 1$

19. Find a function $f(x, y, z)$ whose level surface $f = 1$ is the graph of the function $g(x, y) = x + 2y$.

20. Find two functions $f(x, y)$ and $g(x, y)$ so that the graphs of both together form the ellipsoid $x^2 + y^2/4 + z^2/9 = 1$.

21. Find a formula for a function $g(x, y, z)$ whose level surfaces are planes parallel to the plane $z = 2x + 3y - 5$.

22. The surface S is the graph of $f(x, y) = \sqrt{1 - x^2 - y^2}$.

(a) Explain why S is the upper hemisphere of radius 1, with equator in the xy-plane, centered at the origin.

(b) Find a level surface $g(x, y, z) = c$ representing S.

23. The surface S is the graph of $f(x, y) = \sqrt{1 - y^2}$.

(a) Explain why S is the upper half of a circular cylinder of radius 1, centered along the x-axis.

(b) Find a level surface $g(x, y, z) = c$ representing S.

24. A cone C, with height 1 and radius 1, has its base in the xz-plane and its vertex on the positive y-axis. Find a function $g(x, y, z)$ such that C is part of the level surface $g(x, y, z) = 0$. [Hint: The graph of $f(x, y) = \sqrt{x^2 + y^2}$ is a cone which opens up and has vertex at the origin.]

25. Describe, in words, the level surface $f(x, y, z) = x^2/4 + z^2 = 1$.

26. Describe, in words, the level surface $g(x, y, z) = x^2 + y^2/4 + z^2 = 1$. [Hint: Look at cross-sections with constant x, y and z values.]

27. Describe in words the level surfaces of the function $g(x, y, z) = x + y + z$.

28. Describe in words the level surfaces of $f(x, y, z) = \sin(x + y + z)$.

29. Describe the surface $x^2 + y^2 = (2 + \sin z)^2$. In general, if $f(z) \geq 0$ for all z, describe the surface $x^2 + y^2 = (f(z))^2$.

30. What do the level surfaces of $f(x, y, z) = x^2 - y^2 + z^2$ look like? [Hint: Use cross-sections with y constant instead of cross-sections with z constant.]

31. Describe in words the level surfaces of $g(x, y, z) = e^{-(x^2+y^2+z^2)}$.

32. Sketch and label level surfaces of $h(x, y, z) = e^{z - y}$ for $h = 1, e, e^2$.

33. Sketch and label level surfaces of $f(x, y, z) = 4 - x^2 - y^2 - z^2$ for $f = 0, 1, 2$.

34. Sketch and label level surfaces of $g(x, y, z) = 1 - x^2 - y^2$ for $g = 0, -1, -2$.