12.6 Limits and Continuity

The sheer vertical face of Half Dome, in Yosemite National Park in California, was caused by glacial activity during the Ice Age. (See Figure 12.80.) The height of the terrain rises abruptly by nearly 1000 feet as we scale the rock from the west, whereas it is possible to make a gradual climb to the top from the east.

Figure 12.80 Half Dome in Yosemite National Park

If we consider the function h giving the height of the terrain above sea level in terms of longitude and latitude, then h has a *discontinuity* along the path at the base of the cliff of Half Dome.

Looking at the contour map of the region in Figure 12.81, we see that in most places a small change in position results in a small change in height, except near the cliff. There, no matter how small a step we take, we get a large change in height. (You can see how crowded the contours get near the cliff; some end abruptly along the discontinuity.)

Figure 12.81 A contour map of Half Dome

This geological feature illustrates the ideas of continuity and discontinuity. Roughly speaking, a function is said to be *continuous* at a point if its values at places near the point are close to the value at the point. If this is not the case, the function is said to be *discontinuous*.

The property of continuity is one that, practically speaking, we usually assume of the functions we are studying. Informally, we expect (except under special circumstances) that values of a function do not change drastically when making small changes to the input variables. Whenever we model a one-variable function by an unbroken curve, we are making this assumption. Even when functions come to us as tables of data, we usually make the assumption that the missing function values between data points are close to the measured ones.

In this section we study limits and continuity a bit more formally in the context of functions of several variables. For simplicity we study these concepts for functions of two variables, but our discussion can be adapted to functions of three or more variables.

One can show that sums, products, and compositions of continuous functions are continuous, while the quotient of two continuous functions is continuous everywhere the denominator function is nonzero. Thus, each of the functions

$$\cos(x^2 y), \qquad \ln(x^2+y^2), \qquad \frac{e^{x+y}}{x+y}, \qquad \ln(\sin(x^2+y^2))$$

is continuous at all points (x, y) where it is defined. As for functions of one variable, the graph of a continuous function over an unbroken domain is unbroken—that is, the surface has no holes or rips in it.

Example 1

From Figures 12.82, 12.83, 12.84 and 12.85, which of the following functions appear to be continuous at (0, 0)?

(a)

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(b)

$$g(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

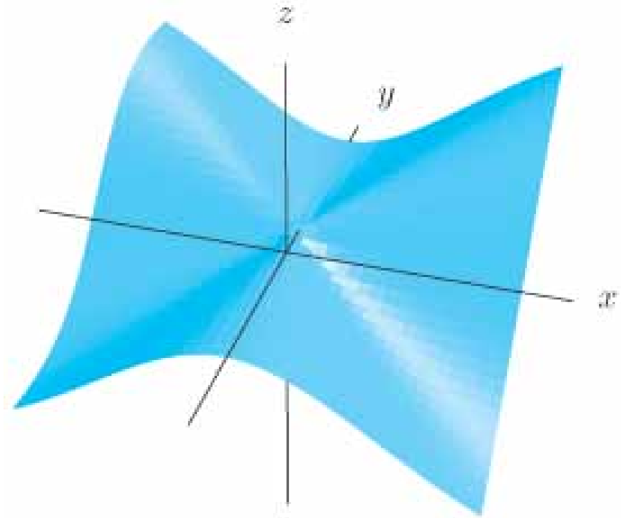

Figure 12.82 Graph of $z = x^2y/(x^2 + y^2)$

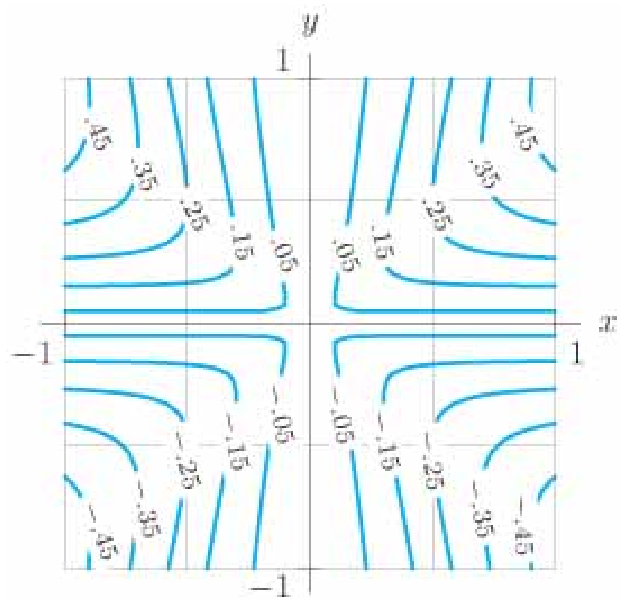

Figure 12.83 Contour diagram of $z = x^2y/(x^2 + y^2)$

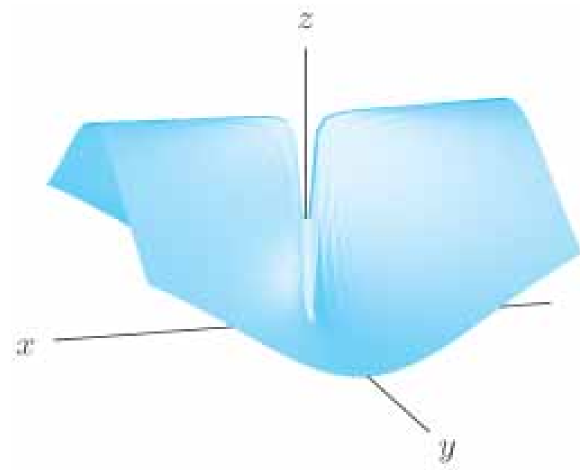

Figure 12.84 Graph of $z = x^2/(x^2 + y^2)$

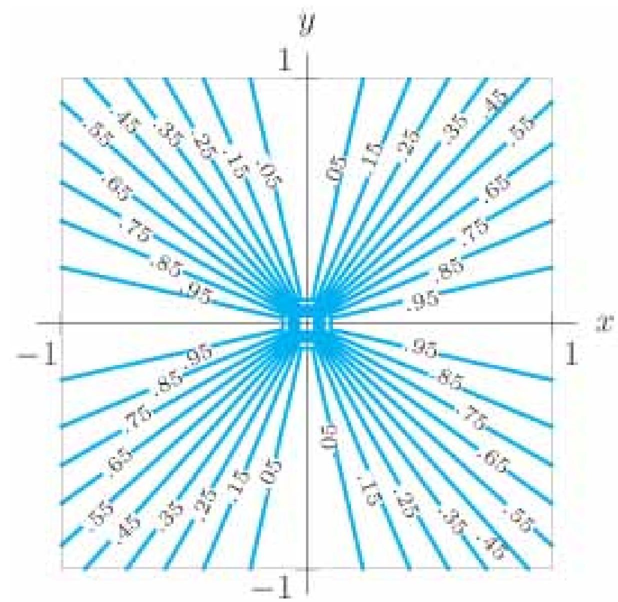

Figure 12.85 Contour diagram of $z = x^2/(x^2 + y^2)$

Solution

(a) The graph and contour diagram of f in Figures 12.82 and 12.83 suggest that f is close to 0 when (x, y) is close to $(0, 0)$. That is, the figures suggest that f is continuous at the point $(0, 0)$; the graph appears to have no rips or holes there.

However, the figures cannot tell us for sure whether f is continuous. To be certain we must investigate the limit analytically, as is done in Example 2(a).

(b) The graph of g and its contours near $(0, 0)$ in Figure 12.84 and 12.85 suggest that g behaves differently from f : The contours of g seem to “crash' at the origin and the graph rises rapidly from 0 to 1 near $(0, 0)$. Small changes in (x, y) near $(0, 0)$ can yield large changes in g, so we expect that g is not continuous at the point $(0, 0)$. Again, a more precise analysis is given in Example 2(b).

The previous example suggests that continuity *at* a point depends on a function's behavior *near* the point. To study behavior near a point more carefully we need the idea of a limit of a function of two variables. Suppose that $f(x, y)$ is a function defined on a set in 2-space, not necessarily containing the point (a, b), but containing points (x, y) arbitrarily close to (a, b); suppose that L is a number.

The function f has a **limit** L at the point (a, b), written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if $f(x, y)$ is as close to L as we please whenever the distance from the point (x, y) to the point (a, b) is sufficiently small, but not zero.

We define continuity for functions of two variables in the same way as for functions of one variable:

A function f is **continuous at the point** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

A function is **continuous on a region** R in the xy-plane if it is continuous at each point in R.

Thus, if f is continuous at the point (a, b), then f must be defined at (a, b) and the limit, $\lim_{(x,y)\to(a,b)} f(x, y)$, must exist and be equal to the value $f(a, b)$. If a function is defined at a point (a, b) but is not continuous there, then we say that f is *discontinuous* at (a, b).

We now apply the definition of continuity to the functions in Example 1, showing that f is continuous at $(0, 0)$ and that g is discontinuous at $(0, 0)$.

Example 2

Let f and g be the functions in Example 1. Use the definition of the limit to show that:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = 0$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist.

Solution

To investigate these limits of f and g, we consider values of these functions near, but not at, the origin, where they are given by the formulas

$$f(x, y) = \frac{x^2 y}{x^2 + y^2} \qquad g(x, y) = \frac{x^2}{x^2 + y^2}.$$

(a) The graph and contour diagram of f both suggest that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. To use the definition of the limit, we estimate $|f(x, y) - L|$ with $L = 0$:

$$|f(x,y) - L| = \left|\frac{x^2 y}{x^2+y^2} - 0\right| = \left|\frac{x^2}{x^2+y^2}\right| |y| \le |y| \le \sqrt{x^2+y^2}.$$

Now $\sqrt{x^2+y^2}$ is the distance from (x, y) to $(0, 0)$. Thus, to make $|f(x, y) - 0| < 0.001$, for example, we need only require (x, y) be within 0.001 of (0, 0). More generally, for any positive number u, no matter how small, we are sure that $|f(x, y) - 0| < u$ whenever (x, y) is no farther than u from (0, 0). This is what we mean by saying that the difference $|f(x, y) - 0|$ can be made as small as we wish by choosing the distance to be sufficiently small. Thus, we conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$$

Notice that since this limit equals $f(0, 0)$, the function f is continuous at (0, 0).

(b) Although the formula defining the function g looks similar to that of f, we saw in Example 1 that g's behavior near the origin is quite different. If we consider points $(x, 0)$ lying along the x-axis near (0, 0), then the values $g(x, 0)$ are equal to 1, while if we consider points $(0, y)$ lying along the y-axis near (0, 0), then the values $g(0, y)$ are equal to 0. Thus, within any distance (no matter how small) from the origin, there are points where $g = 0$ and points where $g = 1$. Therefore the limit $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist, and thus g is not continuous at (0, 0).

While the notions of limit and continuity look formally the same for one- and two-variable functions, they are somewhat more subtle in the multivariable case. The reason for this is that on the line (1-space), we can approach a point from just two directions (left or right) but in 2-space there are an infinite number of ways to approach a given point.

Exercises and Problems for Section 12.6

Exercises

Are the functions in Exercises 1, 2, 3, 4, 5 and 6 continuous at all points in the given regions?

1. $\dfrac{1}{x^2+y^2}$ on the square $-1 \le x \le 1$, $-1 \le y \le 1$

2. $\dfrac{1}{x^2+y^2}$ on the square $1 \le x \le 2$, $1 \le y \le 2$

3. $\dfrac{y}{x^2+2}$ on the disk $x^2 + y^2 \le 1$

4. $\dfrac{e^{\sin x}}{\cos y}$ on the rectangle $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{4}$

5. $\tan(xy)$ on the square $-2 \le x \le 2$, $-2 \le y \le 2$

6. $\sqrt{2x-y}$ on the disk $x^2 + y^2 \le 4$

In Exercises 7, 8, 9, 10 and 11, find the limits of the functions $f(x, y)$ as $(x, y) \to (0, 0)$. Assume that polynomials, exponentials, logarithmic, and trigonometric functions are continuous.

7. $f(x, y) = e^{-x-y}$

8. $f(x, y) = x^2 + y^2$

9. $f(x,y) = \dfrac{x}{x^2+1}$

10. $f(x,y) = \dfrac{x+y}{(\sin y)+2}$

11. $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ [Hint: $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$.]

Problems

In Problems 12 and 13, show that the function $f(x, y)$ does not have a limit as $(x, y) \to (0, 0)$. [Hint: Use the line $y = mx$.]

12. $$f(x,y) = \frac{x+y}{x-y}, \quad x \neq y$$

13. $$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

14. Show that $f(x, y)$ has no limit as $(x, y) \to (0, 0)$ if

$$f(x,y) = \frac{xy}{|xy|}, \quad x \neq 0 \text{ and } y \neq 0.$$

15. Show that the function f does not have a limit at $(0, 0)$ by examining the limits of f as $(x, y) \to (0, 0)$ along the line $y = x$ and along the parabola $y = x^2$:

$$f(x,y) = \frac{x^2 y}{x^4+y^2}, \quad (x,y) \neq (0,0).$$

16. Show that the function f does not have a limit at $(0, 0)$ by examining the limits of f as $(x, y) \to (0, 0)$ along the curve $y = kx^2$ for different values of k:

$$f(x,y) = \frac{x^2}{x^2+y}, \quad x^2 + y \neq 0.$$

17. Explain why the following function is not continuous along the line $y = 0$:

$$f(x,y) = \begin{cases} 1-x, & y \geq 0, \\ -2, & y < 0. \end{cases}$$

In Problems 18 and 19, determine whether there is a value for c making the function continuous everywhere. If so, find it. If not, explain why not.

18. $$f(x,y) = \begin{cases} c+y, & x \leq 3, \\ 5-x, & x > 3. \end{cases}$$

19.

$$f(x,y) = \begin{cases} c+y, & x \le 3, \\ 5-y, & x > 3. \end{cases}$$

20. Is the following function continuous at (0, 0)?

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$$

21. What value of *c* makes the following function continuous at (0, 0)?

$$f(x,y) = \begin{cases} x^2 + y^2 + 1 & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

22. (a) Use a computer to draw the graph and the contour diagram of the following function:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(b) Do your answers to part (a) suggest that *f* is continuous at (0, 0)? Explain your answer.

23. The function *f* , whose graph and contour diagram are in Figures 12.86 and 12.87, is given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Show that *f* (0, *y*) and *f* (*x*, 0) are each continuous functions of one variable.

(b) Show that rays emanating from the origin are contained in contours of *f* .

(c) Is *f* continuous at (0, 0)?

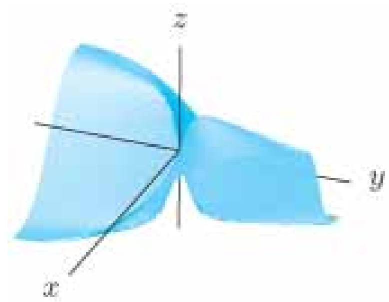

Figure 12.86 Graph of $z = xy/(x^2 + y^2)$

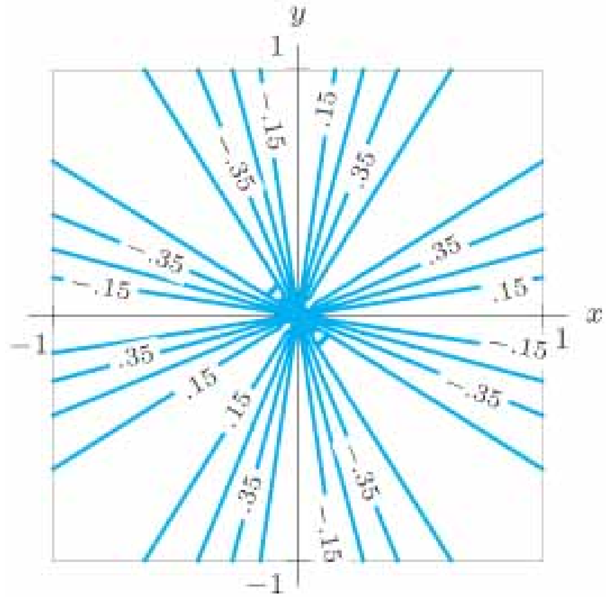

Figure 12.87 Contour diagram of $z = xy/(x^2 + y^2)$