36. For each of the four statements below, answer the following questions: Does the statement make sense? If yes, is it true for all possible choices of $\vec{a}$ and $\vec{b}$? If no, why not?

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ **(b)** $\vec{a} + \|\vec{b}\| = \|\vec{a} + \vec{b}\|$
(c) $\|\vec{b} + \vec{a}\| = \|\vec{a} + \vec{b}\|$ **(d)** $\|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$.

37. Two adjacent sides of a regular hexagon are given as the vectors $\vec{u}$ and $\vec{v}$ in Figure 13.17. Label the remaining sides in terms of $\vec{u}$ and $\vec{v}$.

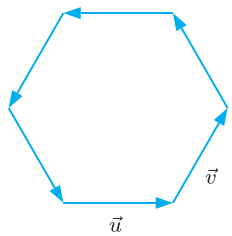

Figure 13.17

38. For what values of t are the following pairs of vectors parallel?

(a) $2\vec{i} + (t^2 + \frac{2}{3}t + 1)\vec{j} + t\vec{k}$, $6\vec{i} + 8\vec{j} + 3\vec{k}$
(b) $t\vec{i} + \vec{j} + (t-1)\vec{k}$, $2\vec{i} - 4\vec{j} + \vec{k}$
(c) $2t\vec{i} + t\vec{j} + t\vec{k}$, $6\vec{i} + 3\vec{j} + 3\vec{k}$.

39. Find all vectors $\vec{v}$ in 2 dimensions having $\|\vec{v}\| = 5$ and the $\vec{i}$-component of $\vec{v}$ is $3\vec{i}$.

40. Find all vectors $\vec{v}$ in the plane such that $\|\vec{v}\| = 1$ and $\|\vec{v} + \vec{i}\| = 1$.

41. A truck is traveling due north at 30 km/hr approaching a crossroad. On a perpendicular road a police car is traveling west toward the intersection at 40 km/hr. Both vehicles will reach the crossroad in exactly one hour. Find the vector currently representing the displacement of the truck with respect to the police car.

42. Figure 13.18 shows a molecule with four atoms at O, A, B and C. Verify that every atom in the molecule is 2 units away from every other atom.

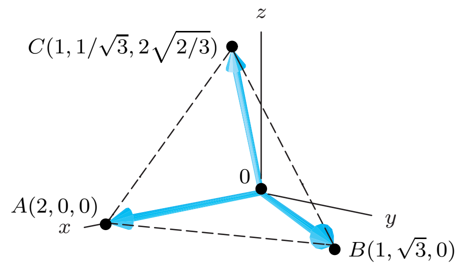

Figure 13.18

43. Show that the medians of a triangle intersect at a point $\frac{1}{3}$ of the way along each median from the side it bisects.

44. In the game of laser tag, you shoot a harmless laser gun and try to hit a target worn at the waist by other players. Suppose you are standing at the origin of a three dimensional coordinate system and that the xy-plane is the floor. Suppose that waist-high is 3 feet above floor level and that eye level is 5 feet above the floor. Three of your friends are your opponents. One is standing so that his target is 30 feet along the x-axis, the other lying down so that his target is at the point $x = 20$, $y = 15$, and the third lying in ambush so that his target is at a point 8 feet above the point $x = 12$, $y = 30$.

(a) If you aim with your gun at eye level, find the vector from your gun to each of the three targets.
(b) If you shoot from waist height, with your gun one foot to the right of the center of your body as you face along the x-axis, find the vector from your gun to each of the three targets.

13.2 VECTORS IN GENERAL

Besides displacement, there are many quantities that have both magnitude and direction and are added and multiplied by scalars in the same way as displacements. Any such quantity is called a *vector* and is represented by an arrow in the same manner we represent displacements. The length of the arrow is the *magnitude* of the vector, and the direction of the arrow is the direction of the vector.

Velocity Versus Speed

The speed of a moving body tells us how fast it is moving, say 80 km/hr. The speed is just a number; it is therefore a scalar. The velocity, on the other hand, tells us both how fast the body is moving and the direction of motion; it is a vector. For instance, if a car is heading northeast at 80 km/hr, then its velocity is a vector of length 80 pointing northeast.

> The **velocity vector** of a moving object is a vector whose magnitude is the speed of the object and whose direction is the direction of its motion.

Example 1 A car is traveling north at a speed of 100 km/hr, while a plane above is flying horizontally south-west at a speed of 500 km/hr. Draw the velocity vectors of the car and the plane.

Solution Figure 13.19 shows the velocity vectors. The plane's velocity vector is five times as long as the car's, because its speed is five times as great.

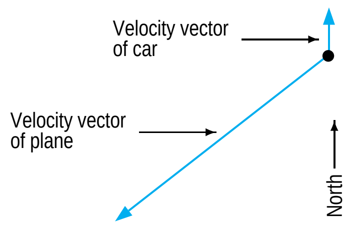

Figure 13.19: Velocity vector of the car is 100 km/hr north and of the plane is 500 km/hr south-west

The next example illustrates that the velocity vectors for two motions add to give the velocity vector for the combined motion, just as displacements do.

Example 2 A riverboat is moving with velocity $\vec{v}$ and a speed of 8 km/hr relative to the water. In addition, the river has a current $\vec{c}$ and a speed of 1 km/hr. (See Figure 13.20.) What is the physical significance of the vector $\vec{v} + \vec{c}$?

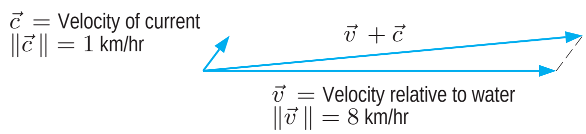

Figure 13.20: Boat's velocity relative to the river bed is the sum, $\vec{v} + \vec{c}$

Solution The vector $\vec{v}$ shows how the boat is moving relative to the water, while $\vec{c}$ shows how the water is moving relative to the riverbed. During an hour, imagine that the boat first moves 8 km relative to the water, which remains still; this displacement is represented by $\vec{v}$. Then imagine the water moving 1 km while the boat remains stationary relative to the water; this displacement is represented by $\vec{c}$. The combined displacement is represented by $\vec{v} + \vec{c}$. Thus, the vector $\vec{v} + \vec{c}$ is the velocity of the boat relative to the riverbed.

Note that the effective speed of the boat is not necessarily 9 km/hr unless the boat is moving in the direction of the current. Although we add the velocity vectors, we do not necessarily add their lengths.

Scalar multiplication also makes sense for velocity vectors. For example, if $\vec{v}$ is a velocity vector, then $-2\vec{v}$ represents a velocity of twice the magnitude in the opposite direction.

Example 3 A ball is moving with velocity $\vec{v}$ when it hits a wall at a right angle and bounces straight back, with its speed reduced by 20%. Express its new velocity in terms of the old one.

Solution The new velocity is $-0.8\vec{v}$, where the negative sign expresses the fact that the new velocity is in the direction opposite to the old.

We can represent velocity vectors in components in the same way we did on page 691.

Example 4 Represent the velocity vectors of the car and the plane in Example 1 using components. Take north to be the positive y-axis, east to be the positive x-axis, and upward to be the positive z-axis.

Solution The car is traveling north at 100 km/hr, so the y-component of its velocity is $100\vec{j}$ and the x-component is $0\vec{i}$. Since it is traveling horizontally, the z-component is $0\vec{k}$. So we have

$$\text{Velocity of car } = 0\vec{i} + 100\vec{j} + 0\vec{k} = 100\vec{j}.$$

The plane's velocity vector also has $\vec{k}$ component equal to zero. Since it is traveling southwest, its $\vec{i}$ and $\vec{j}$ components have negative coefficients (north and east are positive). Since the plane is traveling at 500 km/hr, in one hour it is displaced $500/\sqrt{2} \approx 354$ km to the west and 354 km to the south. (See Figure 13.21.) Thus,

$$\text{Velocity of plane} = -(500\cos 45°)\vec{i} - (500\sin 45°)\vec{j} \approx -354\vec{i} - 354\vec{j}.$$

Of course, if the car were climbing a hill or if the plane were descending for a landing, then the $\vec{k}$ component would not be zero.

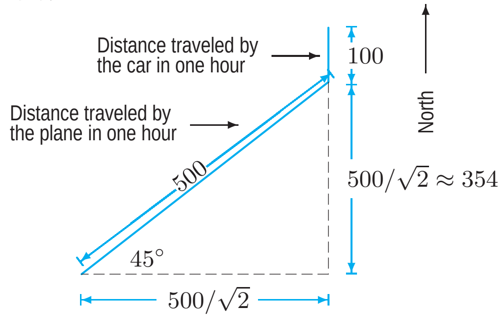

Figure 13.21: Distance traveled by the plane and car in one hour

Acceleration

Another example of a vector quantity is acceleration. Acceleration, like velocity, is specified by both a magnitude and a direction — for example, the acceleration due to gravity is 9.81 m/sec^2 vertically downward.

Force

Force is another example of a vector quantity. Suppose you push on an open door. The result depends both on how hard you push and in what direction. Thus, to specify a force we must give its magnitude (or strength) and the direction in which it is acting. For example, the gravitational force exerted on an object by the earth is a vector pointing from the object toward the center of the earth; its magnitude is the strength of the gravitational force.

Example 5 The earth travels around the sun in an ellipse. The gravitational force on the earth and the velocity of the earth are governed by the following laws:

Newton's Law of Gravitation: The magnitude of the gravitational attraction, F, between two masses m_1 and m_2 at a distance r apart is given by $F = Gm_1m_2/r^2$, where G is a constant. The force vector lies along the line between the masses.

Kepler's Second Law: The line joining a planet to the sun sweeps out equal areas in equal times.

(a) Sketch vectors representing the gravitational force of the sun on the earth at two different positions in the earth's orbit.
(b) Sketch the velocity vector of the earth at two points in its orbit.

Solution

(a) Figure 13.22 shows the earth orbiting the sun. Note that the gravitational force vector always points toward the sun and is larger when the earth is closer to the sun because of the r^2 term in the denominator. (In fact, the real orbit looks much more like a circle than we have shown here.)
(b) The velocity vector points in the direction of motion of the earth. Thus, the velocity vector is tangent to the ellipse. See Figure 13.23. Furthermore, the velocity vector is longer at points of the orbit where the planet is moving quickly, because the magnitude of the velocity vector is the speed. Kepler's Second Law enables us to determine when the earth is moving quickly and

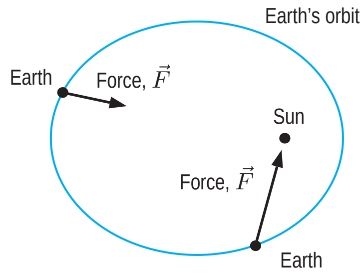

Figure 13.22: Gravitational force, $\vec{F}$, exerted by the sun on the earth: Greater magnitude closer to sun

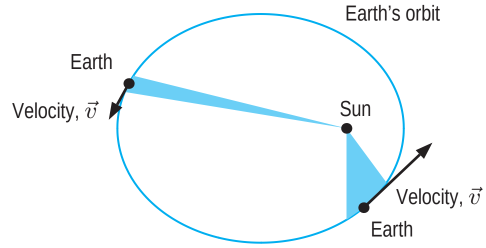

Figure 13.23: The velocity vector, $\vec{v}$, of the earth: Greater magnitude closer to the sun

when it is moving slowly. Over a fixed period of time, say one month, the line joining the earth to the sun sweeps out a sector having a certain area. Figure 13.23 shows two sectors swept out in two different one-month time-intervals. Kepler's law says that the areas of the two sectors are the same. Thus, the earth must move farther in a month when it is close to the sun than when it is far from the sun. Therefore, the earth moves faster when it is closer to the sun and slower when it is farther away.

Properties of Addition and Scalar Multiplication

In general, vectors add, subtract, and are multiplied by scalars in the same way as displacement vectors. Thus, for any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and any scalars α and β, we have the following properties:

Commutativity

1. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$

Associativity

2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

3. $\alpha(\beta\vec{v})=(\alpha\beta)\vec{v}$

Distributivity

4. $(\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v}$

5. $\alpha(\vec{v}+\vec{w})=\alpha\vec{v}+\alpha\vec{w}$

Identity

6. $1\vec{v}=\vec{v}$

7. $0\vec{v}=\vec{0}$

8. $\vec{v}+\vec{0}=\vec{v}$

9. $\vec{w}+(-1)\vec{v}=\vec{w}-\vec{v}$

Problems 28–34 at the end of this section ask for a justification of these results in terms of displacement vectors.

Using Components

Example 6 A plane, heading due east at an airspeed of 600 km/hr, experiences a wind of 50 km/hr blowing toward the northeast. Find the plane's direction and ground speed.

Solution We choose a coordinate system with the x-axis pointing east and the y-axis pointing north. See Figure 13.24.

The airspeed tells us the speed of the plane relative to still air. Thus, the plane is moving due east with velocity $\vec{v}=600\vec{i}$ relative to still air. In addition, the air is moving with a velocity $\vec{w}$. Writing $\vec{w}$ in components, we have

$$\vec{w}=(50\cos 45^\circ)\vec{i}+(50\sin 45^\circ)\vec{j}=35.4\vec{i}+35.4\vec{j}.$$

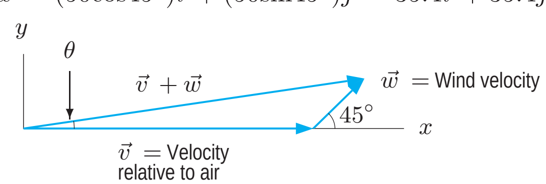

Figure 13.24: Plane's velocity relative to the ground is the sum $\vec{v}+\vec{w}$

The vector $\vec{v} + \vec{w}$ represents the displacement of the plane in one hour relative to the ground. Therefore, $\vec{v} + \vec{w}$ is the velocity of the plane relative to the ground. In components, we have

$$\vec{v} + \vec{w} = 600\vec{i} + \left(35.4\vec{i} + 35.4\vec{j}\right) = 635.4\vec{i} + 35.4\vec{j}.$$

The direction of the plane's motion relative to the ground is given by the angle θ in Figure 13.24, where

$$\tan\theta = \frac{35.4}{635.4}$$

so

$$\theta = \arctan\left(\frac{35.4}{635.4}\right) = 3.2^\circ.$$

The ground speed is the speed of the plane relative to the ground, so

$$\text{Groundspeed} = \sqrt{635.4^2 + 35.4^2} = 636.4 \text{ km/hr}.$$

Thus, the speed of the plane relative to the ground has been increased slightly by the wind. (This is as we would expect, as the wind has a positive component in the direction in which the plane is traveling.) The angle θ shows how far the plane is blown off course by the wind.

Vectors in n Dimensions

Using the alternative notation $\vec{v} = (v_1, v_2, v_3)$ for a vector in 3-space, we can define a vector in n dimensions as a string of n numbers. Thus, a vector in n dimensions can be written as

$$\vec{c} = (c_1, c_2, \ldots, c_n).$$

Addition and scalar multiplication are defined by the formulas

$$\vec{v} + \vec{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

and

$$\lambda\vec{v} = \lambda(v_1, v_2, \ldots, v_n) = (\lambda v_1, \lambda v_2, \ldots, \lambda v_n).$$

Why Do We Want Vectors in n Dimensions?

Vectors in two and three dimensions can be used to model displacement, velocities, or forces. But what about vectors in n dimensions? There is another interpretation of 3-dimensional vectors (or 3-vectors) which is useful: they can be thought of as listing 3 different quantities — for example, the displacements parallel to the x, y, and z axes. Similarly, the n-vector

$$\vec{c} = (c_1, c_2, \ldots, c_n)$$

can be thought of as a way of keeping n different quantities organized. For example, a *population* vector $\vec{N}$ shows the number of children and adults in a population:

$$\vec{N} = (\text{Number of children}, \text{ Number of adults}),$$

or, if we are interested in a more detailed breakdown of ages, we might give the number in each ten-year age bracket in the population (up to age 110) in the form

$$\vec{N} = (N_1, N_2, N_3, N_4, \ldots, N_{10}, N_{11}),$$

where N_1 is the population aged 0–9, and N_2 is the population aged 10–19, and so on.

A *consumption* vector,

$$\vec{q} = (q_1, q_2, \ldots, q_n)$$

shows the quantities $q_1, q_2, \ldots, q_n$ consumed of each of n different goods. A *price* vector

$$\vec{p} = (p_1, p_2, \ldots, p_n)$$

contains the prices of n different items.

In 1907, Hermann Minkowski used vectors with four components when he introduced *space-time coordinates*, whereby each event is assigned a vector position $\vec{v}$ with four coordinates, three for its position in space and one for time:

$$\vec{v} = (x, y, z, t).$$

Example 7 Suppose the vector $\vec{I}$ represents the number of copies, in thousands, made by each of four copy centers in the month of December and $\vec{J}$ represents the number of copies made at the same four copy centers during the previous eleven months (the "year-to-date"). If $\vec{I} = (25, 211, 818, 642)$, and $\vec{J} = (331, 3227, 1377, 2570)$, compute $\vec{I} + \vec{J}$. What does this sum represent?

Solution The sum is

$$\vec{I} + \vec{J} = (25 + 331, 211 + 3227, 818 + 1377, 642 + 2570) = (356, 3438, 2195, 3212).$$

Each term in $\vec{I} + \vec{J}$ represents the sum of the number of copies made in December plus those in the previous eleven months, that is, the total number of copies made during the entire year at that particular copy center.

Example 8 The price vector $\vec{p} = (p_1, p_2, p_3)$ represents the prices in dollars of three goods. Write a vector which gives the prices of the same goods in cents.

Solution The prices in cents are $100p_1$, $100p_2$, and $100p_3$ respectively, so the new price vector is

$$(100p_1, 100p_2, 100p_3) = 100\vec{p}.$$

Exercises and Problems for Section 13.2

Exercises

In Exercises 1–5, say whether the given quantity is a vector or a scalar.

1. The population of the US.

2. The distance from Seattle to St. Louis.

3. The temperature at a point on the earth's surface.

4. The magnetic field at a point on the earth's surface.

5. The populations of each of the 50 states.

6. A car is traveling at a speed of 50 km/hr. The positive y-axis is north and the positive x-axis is east. Resolve the car's velocity vector (in 2-space) into components if the car is traveling in each of the following directions:

(a) East **(b)** South
(c) Southeast **(d)** Northwest.

7. Give the components of the velocity vector for wind blowing at 10 km/hr toward the southeast. (Assume north is the positive y-direction.)

8. Give the components of the velocity vector of a boat which is moving at 40 km/hr in a direction 20° south of west. (Assume north is in the positive y-direction.)

9. Which is traveling faster, a car whose velocity vector is $21\vec{i} + 35\vec{j}$, or a car whose velocity vector is $40\vec{i}$, assuming that the units are the same for both directions?

10. What angle does a force of $\vec{F} = 15\vec{i} + 18\vec{j}$ make with the x-axis?

Problems

11. The velocity of the current in a river is $\vec{c} = 0.6\vec{i} + 0.8\vec{j}$ km/hr. A boat moves relative to the water with velocity $\vec{v} = 8\vec{i}$ km/hr.

(a) What is the speed of the boat relative to the riverbed?
(b) What angle does the velocity of the boat relative to the riverbed make with the vector $\vec{v}$? What does this angle tell us in practical terms?

12. Suppose the current in Problem 11 is twice as fast and in the opposite direction. What is the speed of the boat with respect to the riverbed?

13. A plane is heading due east and climbing at the rate of 80 km/hr. If its airspeed is 480 km/hr and there is a wind blowing 100 km/hr to the northeast, what is the ground speed of the plane?

14. An airplane is flying at an airspeed of 500 km/hr in a wind blowing at 60 km/hr toward the southeast. In what direction should the plane head to end up going due east? What is the airplane's speed relative to the ground?

15. An airplane is flying at an airspeed of 600 km/hr in a cross-wind that is blowing from the northeast at a speed of 50 km/hr. In what direction should the plane head to end up going due east?

16. The current in a river is pushing a boat in direction 25° north of east with a speed of 12 km/hr. The wind is pushing the same boat in a direction 80° south of east with a speed of 7 km/hr. Find the velocity vector of the boat's engine (relative to the water) if the boat actually moves due east at a speed of 40 km/hr relative to the ground.

17. A large ship is being towed by two tugs. The larger tug exerts a force which is 25% greater than the smaller tug and at an angle of 30 degrees north of east. Which direction must the smaller tug pull to ensure that the ship travels due east?

18. A particle moving with speed v hits a barrier at an angle of 60° and bounces off at an angle of 60° in the opposite direction with speed reduced by 20 percent. See Figure 13.25. Find the velocity vector of the object after impact.

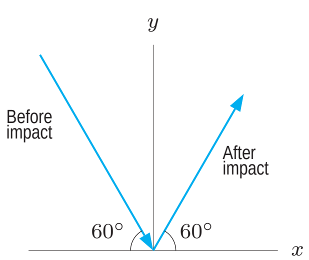

Figure 13.25

19. There are five students in a class. Their scores on the midterm (out of 100) are given by the vector $\vec{v} = (73, 80, 91, 65, 84)$. Their scores on the final (out of 100) are given by $\vec{w} = (82, 79, 88, 70, 92)$. If the final counts twice as much as the midterm, find a vector giving the total scores (as a percentage) of the students.

20. The price vector of beans, rice, and tofu is $(0.30, 0.20, 0.50)$ in dollars per pound. Express it in dollars per ounce.

21. Two forces, represented by the vectors $\vec{F}_1 = 8\vec{i} - 6\vec{j}$ and $\vec{F}_2 = 3\vec{i} + 2\vec{j}$, are acting on an object. Give a vector representing the force that must be applied to the object if it is to remain stationary.

22. One force is pushing an object in a direction 50° south of east with a force of 25 newtons. A second force is simultaneously pushing the object in a direction 70° north of west with a force of 60 newtons. If the object is to remain stationary, give the direction and magnitude of the third force which must be applied to the object to counterbalance the first two.

23. An object P is pulled by a force $\vec{F}_1$ of magnitude 15 lb at an angle of 20 degrees north of east. In what direction must a force $\vec{F}_2$ of magnitude 20 lb pull to ensure that P moves due east?

24. An airplane heads northeast at an airspeed of 700 km/hr, but there is a wind blowing from the west at 60 km/hr. In what direction does the plane end up flying? What is its speed relative to the ground?

25. A man wishes to row the shortest possible distance from north to south across a river which is flowing at 4 km/hr from the east. He can row at 5 km/hr.

(a) In which direction should he steer?
(b) If there is a wind of 10 km/hr from the southwest, in which direction should he steer to try and go directly across the river? What happens?

26. An object is moving counterclockwise at a constant speed around the circle $x^2 + y^2 = 1$, where x and y are measured in meters. It completes one revolution every minute.

(a) What is its speed?
(b) What is its velocity vector 30 seconds after it passes the point $(1, 0)$? Does your answer change if the object is moving clockwise? Explain.

27. An object is attached by a string to a fixed point and rotates 30 times per minute in a horizontal plane. Show that the speed of the object is constant but the velocity is not. What does this imply about the acceleration?

Use the geometric definition of addition and scalar multiplication to explain each of the properties in Problems 28–35.

28. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$

29. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

30. $\alpha(\vec{v}+\vec{w}) = \alpha\vec{v}+\alpha\vec{w}$

31. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

32. $\vec{v}+\vec{0} = \vec{v}$

33. $1\vec{v} = \vec{v}$

34. $\vec{v}+(-1)\vec{w} = \vec{v}-\vec{w}$

35. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

36. The earth is at the origin, the moon is at the point $(384, 0)$, and a spaceship is at $(280, 90)$, where distance is in thousands of kilometers.

(a) What is the displacement vector of the moon relative to the earth? Of the spaceship relative to the earth? Of the spaceship relative to the moon?

(b) How far is the spaceship from the earth? From the moon?

(c) The gravitational force on the spaceship from the earth is 461 newtons and from the moon is 26 newtons. What is the resulting force?

13.3 THE DOT PRODUCT

We have seen how to add vectors; can we multiply two vectors together? In the next two sections we will see two different ways of doing so: the *scalar product* (or *dot product*) which produces a scalar, and the *vector product* (or *cross product*), which produces a vector.

Definition of the Dot Product

The dot product links geometry and algebra. We already know how to calculate the length of a vector from its components; the dot product gives us a way of computing the angle between two vectors. For any two vectors $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ and $\vec{w} = w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$, shown in Figure 13.26, we define a scalar as follows:

The following two definitions of the **dot product**, or **scalar product**, $\vec{v}\cdot\vec{w}$, are equivalent:

- **Geometric definition**
  $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$ and $0 \le \theta \le \pi$.
- **Algebraic definition**
  $\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3.$

Notice that the dot product of two vectors is a *number*.

Why don't we give just one definition of $\vec{v}\cdot\vec{w}$? The reason is that both definitions are equally important; the geometric definition gives us a picture of what the dot product means and the algebraic definition gives us a way of calculating it.

How do we know the two definitions are equivalent — that is, they really do define the same thing? First, we observe that the two definitions give the same result in a particular example. Then we show why they are equivalent in general.

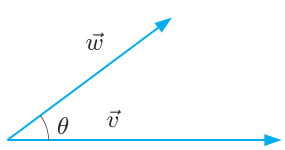

Figure 13.26: The vectors $\vec{v}$ and $\vec{w}$

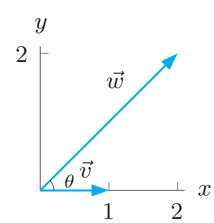

Figure 13.27: Calculating the dot product of the vectors $v = \vec{i}$ and $\vec{w} = 2\vec{i}+2\vec{j}$ geometrically and algebraically gives the same result

Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i}+2\vec{j}$. Compute $\vec{v}\cdot\vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or 45°, and the lengths of the vectors are given by

$$\|\vec{v}\| = 1 \quad\text{and}\quad \|\vec{w}\| = 2\sqrt{2}.$$