30. $\alpha(\vec{v}+\vec{w}) = \alpha\vec{v}+\alpha\vec{w}$

31. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

32. $\vec{v}+\vec{0} = \vec{v}$

33. $1\vec{v} = \vec{v}$

34. $\vec{v}+(-1)\vec{w} = \vec{v}-\vec{w}$

35. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

36. The earth is at the origin, the moon is at the point $(384, 0)$, and a spaceship is at $(280, 90)$, where distance is in thousands of kilometers.

(a) What is the displacement vector of the moon relative to the earth? Of the spaceship relative to the earth? Of the spaceship relative to the moon?

(b) How far is the spaceship from the earth? From the moon?

(c) The gravitational force on the spaceship from the earth is 461 newtons and from the moon is 26 newtons. What is the resulting force?

13.3 THE DOT PRODUCT

We have seen how to add vectors; can we multiply two vectors together? In the next two sections we will see two different ways of doing so: the *scalar product* (or *dot product*) which produces a scalar, and the *vector product* (or *cross product*), which produces a vector.

Definition of the Dot Product

The dot product links geometry and algebra. We already know how to calculate the length of a vector from its components; the dot product gives us a way of computing the angle between two vectors. For any two vectors $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ and $\vec{w} = w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$, shown in Figure 13.26, we define a scalar as follows:

The following two definitions of the **dot product**, or **scalar product**, $\vec{v}\cdot\vec{w}$, are equivalent:

- **Geometric definition**
  $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$ and $0 \le \theta \le \pi$.
- **Algebraic definition**
  $\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3.$

Notice that the dot product of two vectors is a *number*.

Why don't we give just one definition of $\vec{v}\cdot\vec{w}$? The reason is that both definitions are equally important; the geometric definition gives us a picture of what the dot product means and the algebraic definition gives us a way of calculating it.

How do we know the two definitions are equivalent — that is, they really do define the same thing? First, we observe that the two definitions give the same result in a particular example. Then we show why they are equivalent in general.

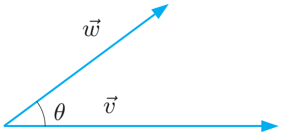

Figure 13.26: The vectors $\vec{v}$ and $\vec{w}$

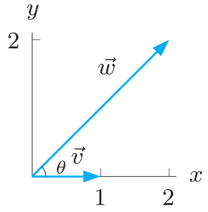

Figure 13.27: Calculating the dot product of the vectors $v = \vec{i}$ and $\vec{w} = 2\vec{i}+2\vec{j}$ geometrically and algebraically gives the same result

Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i}+2\vec{j}$. Compute $\vec{v}\cdot\vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or 45°, and the lengths of the vectors are given by

$$\|\vec{v}\| = 1 \quad \text{and} \quad \|\vec{w}\| = 2\sqrt{2}.$$

Thus,

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta = 1\cdot 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 2.$$

Using the algebraic definition, we get the same result:

$$\vec{v}\cdot\vec{w} = 1\cdot 2 + 0\cdot 2 = 2.$$

Why the Two Definitions of the Dot Product Give the Same Result

In the previous example, the two definitions give the same value for the dot product. To show that the geometric and algebraic definitions of the dot product always give the same result, we must show that, for any vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ with an angle θ between them:

$$\|\vec{v}\|\|\vec{w}\|\cos\theta = v_1w_1 + v_2w_2 + v_3w_3.$$

One method follows; a method which does not use trigonometry is given in Problem 62 on page 710.

Using the Law of Cosines. Suppose that $0 < \theta < \pi$, so that the vectors $\vec{v}$ and $\vec{w}$ form a triangle. (See Figure 13.28.) By the Law of Cosines, we have

$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta.$$

This result is also true for $\theta = 0$ and $\theta = \pi$. We calculate the lengths using components:

$$\begin{aligned}\|\vec{v}\|^2 &= v_1^2 + v_2^2 + v_3^2\\ \|\vec{w}\|^2 &= w_1^2 + w_2^2 + w_3^2\\ \|\vec{v}-\vec{w}\|^2 &= (v_1-w_1)^2 + (v_2-w_2)^2 + (v_3-w_3)^2\\ &= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 + v_3^2 - 2v_3w_3 + w_3^2.\end{aligned}$$

Substituting into the Law of Cosines and canceling, we see that

$$-2v_1w_1 - 2v_2w_2 - 2v_3w_3 = -2\|\vec{v}\|\|\vec{w}\|\cos\theta.$$

Therefore we have the result we wanted, namely that:

$$v_1w_1 + v_2w_2 + v_3w_3 = \|\vec{v}\|\|\vec{w}\|\cos\theta.$$

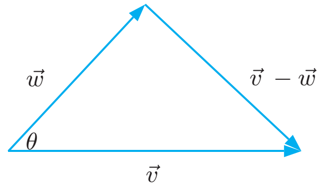

Figure 13.28: Triangle used in the justification of $\|\vec{v}\|\|\vec{w}\|\cos\theta = v_1w_1 + v_2w_2 + v_3w_3$

Properties of the Dot Product

The following properties of the dot product can be justified using the algebraic definition; see Problem 56 on page 709. For a geometric interpretation of Property 3, see Problem 59.

Properties of the Dot Product. For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and any scalar λ,

1. $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
2. $\vec{v}\cdot(\lambda\vec{w}) = \lambda(\vec{v}\cdot\vec{w}) = (\lambda\vec{v})\cdot\vec{w}$
3. $(\vec{v}+\vec{w})\cdot\vec{u} = \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u}$

Perpendicularity, Magnitude, and Dot Products

Two vectors are perpendicular if the angle between them is $\pi/2$ or 90°. Since $\cos(\pi/2) = 0$, if $\vec{v}$ and $\vec{w}$ are perpendicular, then $\vec{v} \cdot \vec{w} = 0$. Conversely, provided that $\vec{v} \cdot \vec{w} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$ and the vectors are perpendicular. Thus, we have the following result:

Two nonzero vectors $\vec{v}$ and $\vec{w}$ are **perpendicular**, or **orthogonal**, if and only if

$$\vec{v} \cdot \vec{w} = 0.$$

For example: $\vec{i} \cdot \vec{j} = 0, \vec{j} \cdot \vec{k} = 0, \vec{i} \cdot \vec{k} = 0$.

If we take the dot product of a vector with itself, then $\theta = 0$ and $\cos\theta = 1$. For any vector $\vec{v}$:

Magnitude and dot product are related as follows:

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2.$$

For example: $\vec{i} \cdot \vec{i} = 1, \vec{j} \cdot \vec{j} = 1, \vec{k} \cdot \vec{k} = 1$.

Using the Dot Product

Depending on the situation, one definition of the dot product may be more convenient to use than the other. In Example 2 which follows, the geometric definition is the only one which can be used because we are not given components. In Example 3, the algebraic definition is used.

Example 2 Suppose the vector $\vec{b}$ is fixed and has length 2; the vector $\vec{a}$ is free to rotate and has length 3. What are the maximum and minimum values of the dot product $\vec{a} \cdot \vec{b}$ as the vector $\vec{a}$ rotates through all possible positions? What positions of $\vec{a}$ and $\vec{b}$ lead to these values?

Solution The geometric definition gives $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = 3 \cdot 2\cos\theta = 6\cos\theta$. Thus, the maximum value of $\vec{a} \cdot \vec{b}$ is 6, and it occurs when $\cos\theta = 1$ so $\theta = 0$, that is, when $\vec{a}$ and $\vec{b}$ point in the same direction. The minimum value of $\vec{a} \cdot \vec{b}$ is -6, and it occurs when $\cos\theta = -1$ so $\theta = \pi$, that is, when $\vec{a}$ and $\vec{b}$ point in opposite directions. (See Figure 13.29.)

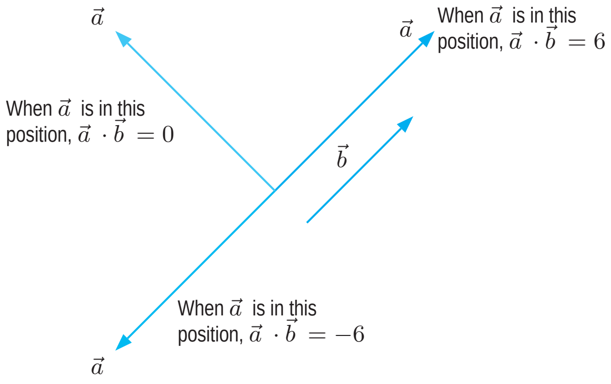

Figure 13.29: Maximum and minimum values of $\vec{a} \cdot \vec{b}$ obtained from a fixed vector $\vec{b}$ of length 2 and rotating vector $\vec{a}$ of length 3

Example 3 Which pairs from the following list of 3-dimensional vectors are perpendicular to one another?

$$\vec{u} = \vec{i} + \sqrt{3}\,\vec{k}, \quad \vec{v} = \vec{i} + \sqrt{3}\,\vec{j}, \quad \vec{w} = \sqrt{3}\,\vec{i} + \vec{j} - \vec{k}.$$

Solution The geometric definition tells us that two vectors are perpendicular if and only if their dot product is zero. Since the vectors are given in components, we calculate dot products using the algebraic definition:

$$\begin{aligned}\vec{v}\cdot\vec{u} &= (\vec{i} + \sqrt{3}\vec{j} + 0\vec{k})\cdot(\vec{i} + 0\vec{j} + \sqrt{3}\vec{k}) = 1\cdot 1 + \sqrt{3}\cdot 0 + 0\cdot\sqrt{3} = 1,\\ \vec{v}\cdot\vec{w} &= (\vec{i} + \sqrt{3}\vec{j} + 0\vec{k})\cdot(\sqrt{3}\vec{i} + \vec{j} - \vec{k}) = 1\cdot\sqrt{3} + \sqrt{3}\cdot 1 + 0(-1) = 2\sqrt{3},\\ \vec{w}\cdot\vec{u} &= (\sqrt{3}\vec{i} + \vec{j} - \vec{k})\cdot(\vec{i} + 0\vec{j} + \sqrt{3}\vec{k}) = \sqrt{3}\cdot 1 + 1\cdot 0 + (-1)\cdot\sqrt{3} = 0.\end{aligned}$$

So the only two vectors which are perpendicular are $\vec{w}$ and $\vec{u}$.

Normal Vectors and the Equation of a Plane

In Section 12.4 we wrote the equation of a plane given its x-slope, y-slope and z-intercept. Now we write the equation of a plane using a vector and a point lying on the plane. A *normal vector* to a plane is a vector that is perpendicular to the plane, that is, it is perpendicular to every displacement vector between any two points in the plane. Let $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ be a normal vector to the plane, let $P_0 = (x_0, y_0, z_0)$ be a fixed point in the plane, and let $P = (x, y, z)$ be any other point in the plane. Then $\overrightarrow{P_0P} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$ is a vector whose head and tail both lie in the plane. (See Figure 13.30.) Thus, the vectors $\vec{n}$ and $\overrightarrow{P_0P}$ are perpendicular, so $\vec{n}\cdot\overrightarrow{P_0P} = 0$. The algebraic definition of the dot product gives $\vec{n}\cdot\overrightarrow{P_0P} = a(x - x_0) + b(y - y_0) + c(z - z_0)$, so we obtain the following result:

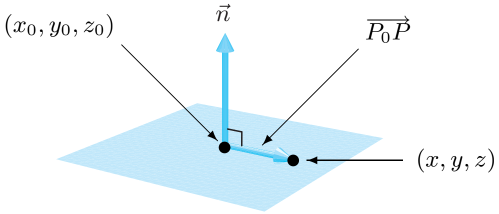

Figure 13.30: Plane with normal $\vec{n}$ and containing a fixed point (x_0, y_0, z_0)

The **equation of the plane** with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P_0 = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Letting $d = ax_0 + by_0 + cz_0$ (a constant), we can write the equation of the plane in the form

$$ax + by + cz = d.$$

Example 4 Find the equation of the plane perpendicular to $-\vec{i} + 3\vec{j} + 2\vec{k}$ and passing through the point $(1, 0, 4)$.

Solution The equation of the plane is

$$-(x - 1) + 3(y - 0) + 2(z - 4) = 0,$$

which simplifies to

$$-x + 3y + 2z = 7.$$

Example 5 Find a normal vector to the plane with equation (a) $x-y+2z=5$ (b) $z=0.5x+1.2y$.

Solution

(a) Since the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$ in a normal vector are the coefficients of x, y, z in the equation of the plane, a normal vector is $\vec{n}=\vec{i}-\vec{j}+2\vec{k}$.

(b) Before we can find a normal vector, we rewrite the equation of the plane in the form

$$0.5x+1.2y-z=0.$$

Thus, a normal vector is $\vec{n}=0.5\vec{i}+1.2\vec{j}-\vec{k}$.

The Dot Product in n Dimensions

The algebraic definition of the dot product can be extended to vectors in higher dimensions.

If $\vec{u}=(u_1,\ldots,u_n)$ and $\vec{v}=(v_1,\ldots,v_n)$ then the dot product of $\vec{u}$ and $\vec{v}$ is the **scalar**

$$\vec{u}\cdot\vec{v}=u_1v_1+\ldots+u_nv_n.$$

Example 6 A video store sells videos, tapes, CDs, and computer games. We define the quantity vector $\vec{q}=(q_1,q_2,q_3,q_4)$, where q_1, q_2, q_3, q_4 denote the quantities sold of each of the items, and the price vector $\vec{p}=(p_1,p_2,p_3,p_4)$, where p_1, p_2, p_3, p_4 denote the price per unit of each item. What does the dot product $\vec{p}\cdot\vec{q}$ represent?

Solution

The dot product is $\vec{p}\cdot\vec{q}=p_1q_1+p_2q_2+p_3q_3+p_4q_4$. The quantity p_1q_1 represents the revenue received by the store for the videos, p_2q_2 represents the revenue for the tapes, and so on. The dot product represents the total revenue received by the store for the sale of these four items.

Resolving a Vector into Components: Projections

In Section 13.1, we resolved a vector into components parallel to the axes. Now we see how to resolve a vector, $\vec{v}$, into components, called $\vec{v}_{\text{parallel}}$ and $\vec{v}_{\text{perp}}$, which are parallel and perpendicular, respectively, to a given nonzero vector, $\vec{u}$. (See Figure 13.31.)

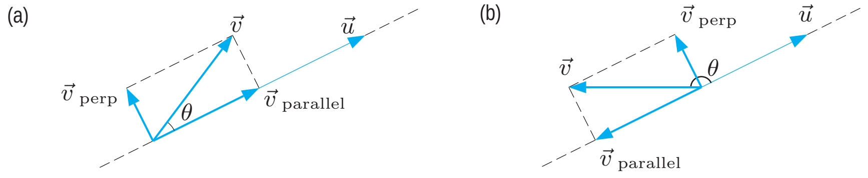

Figure 13.31: Resolving $\vec{v}$ into components parallel and perpendicular to $\vec{u}$
(a) $0<\theta<\pi/2$ (b) $\pi/2<\theta<\pi$

The projection of $\vec{v}$ on $\vec{u}$, written $\vec{v}_{\text{parallel}}$, measures (in some sense) how much the vector $\vec{v}$ is aligned with the vector $\vec{u}$. The length of $\vec{v}_{\text{parallel}}$ is the length of the shadow cast by $\vec{v}$ on a line in the direction of $\vec{u}$.

To compute $\vec{v}_{\text{parallel}}$, we assume $\vec{u}$ is a unit vector. (If not, create one by dividing by its length.) Then Figure 13.31(a) shows that, if $0\le\theta\le\pi/2$:

$$\|\vec{v}_{\text{parallel}}\|=\|\vec{v}\|\cos\theta=\vec{v}\cdot\vec{u}\qquad(\text{since }\|\vec{u}\|=1).$$

Now $\vec{v}_{\text{parallel}}$ is a scalar multiple of $\vec{u}$, and since $\vec{u}$ is a unit vector,

$$\vec{v}_{\text{parallel}}=(\|\vec{v}\|\cos\theta)\vec{u}=(\vec{v}\cdot\vec{u})\vec{u}.$$

A similar argument shows that if $\pi/2 < \theta \leq \pi$, as in Figure 13.31(b), this formula for $\vec{v}_{\text{parallel}}$ still holds. The vector $\vec{v}_{\text{perp}}$ is specified by

$$\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}.$$

Thus, we have the following results:

Projection of $\vec{v}$ on the Line in the Direction of the Unit Vector $\vec{u}$

If $\vec{v}_{\text{parallel}}$ and $\vec{v}_{\text{perp}}$ are components of $\vec{v}$ which are parallel and perpendicular, respectively, to $\vec{u}$, then

$$\begin{matrix}\text{Projection of} \\ \vec{v} \text{ on to } \vec{u}\end{matrix} = \vec{v}_{\text{parallel}} = (\vec{v} \cdot \vec{u})\vec{u} \qquad \text{provided } \|\vec{u}\| = 1$$

$$\text{and} \qquad \vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\text{perp}} \qquad \text{so} \qquad \vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}.$$

Example 7 Figure 13.32 shows the force the wind exerts on the sail of a sailboat. Find the component of the force in the direction in which the sailboat is traveling.

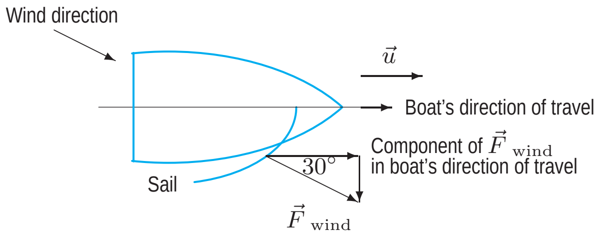

Figure 13.32: Wind moving a sailboat

Solution Let $\vec{u}$ be a unit vector in the direction of travel. The force of the wind on the sail makes an angle of 30° with $\vec{u}$. Thus, the component of this force in the direction of $\vec{u}$ is

$$\vec{F}_{\text{parallel}} = (\vec{F} \cdot \vec{u})\vec{u} = \|\vec{F}\|(\cos 30^\circ)\vec{u} = 0.87\|\vec{F}\|\vec{u}.$$

Thus, the boat is being pushed forward with about 87% of the total force due to the wind. (In fact, the interaction of wind and sail is much more complex than this model suggests.)

A Physical Interpretation of the Dot Product: Work

In physics, the word "work" has a slightly different meaning from its everyday meaning. In physics, when a force of magnitude F acts on an object through a distance d, we say the *work*, W, done by the force is

$$W = Fd,$$

provided the force and the displacement are in the same direction. For example, if a 1 kg body falls 10 meters under the force of gravity, which is 9.8 newtons, then the work done by gravity is

$$W = (9.8 \text{ newtons}) \cdot (10 \text{ meters}) = 98 \text{ joules}.$$

What if the force and the displacement are not in the same direction? Suppose a force $\vec{F}$ acts on an object as it moves along a displacement vector $\vec{d}$. Let θ be the angle between $\vec{F}$ and $\vec{d}$. First, we assume $0 \leq \theta \leq \pi/2$. Figure 13.33 shows how we can resolve $\vec{F}$ into components that are parallel and perpendicular to $\vec{d}$:

$$\vec{F} = \vec{F}_{\text{parallel}} + \vec{F}_{\text{perp}},$$

Then the work done by $\vec{F}$ is defined to be

$$W = \|\vec{F}_{\text{parallel}}\| \, \|\vec{d}\|.$$

We see from Figure 13.33 that $\vec{F}_{\text{parallel}}$ has magnitude $\|\vec{F}\| \cos\theta$. So the work is given by the dot product:

$$W = (\|\vec{F}\| \cos\theta)\|\vec{d}\| = \|\vec{F}\|\|\vec{d}\| \cos\theta = \vec{F} \cdot \vec{d}.$$

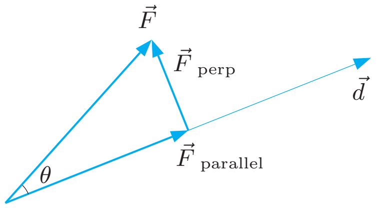

Figure 13.33: Resolving the force $\vec{F}$ into two forces, one parallel to $\vec{d}$, one perpendicular to $\vec{d}$

The formula $W = \vec{F} \cdot \vec{d}$ holds when $\pi/2 < \theta \leq \pi$ also. In that case, the work done by the force is negative and the object is moving against the force. Thus, we have the following definition:

> The **work**, W, done by a force $\vec{F}$ acting on an object through a displacement $\vec{d}$ is given by
>
> $$W = \vec{F} \cdot \vec{d}.$$

Notice that if the vectors $\vec{F}$ and $\vec{d}$ are parallel and in the same direction, with magnitudes F and d, then $\cos\theta = \cos 0 = 1$, so $W = \|\vec{F}\|\|\vec{d}\| = Fd$, which is the original definition. When the vectors are perpendicular, $\cos\theta = \cos\frac{\pi}{2} = 0$, so $W = 0$ and no work is done in the technical definition of the word. For example, if you carry a heavy box across the room at the same horizontal height, no work is done by gravity because the force of gravity is vertical but the motion is horizontal.

Exercises and Problems for Section 13.3

Exercises

For Exercises 1–9, perform the following operations on the given 3-dimensional vectors.

$$\vec{a} = 2\vec{j} + \vec{k} \qquad \vec{b} = -3\vec{i} + 5\vec{j} + 4\vec{k} \qquad \vec{c} = \vec{i} + 6\vec{j}$$
$$\vec{y} = 4\vec{i} - 7\vec{j} \qquad \vec{z} = \vec{i} - 3\vec{j} - \vec{k}$$

1. $\vec{a} \cdot \vec{y}$

2. $\vec{c} \cdot \vec{y}$

3. $\vec{a} \cdot \vec{b}$

4. $\vec{a} \cdot \vec{z}$

5. $\vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y}$

6. $\vec{a} \cdot (\vec{c} + \vec{y})$

7. $(\vec{a} \cdot \vec{b})\vec{a}$

8. $(\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z})$

9. $((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a}$

In Exercises 10–14, find a normal vector to the plane.

10. $2x + y - z = 23$

11. $1.5x + 3.2y + z = 0$

12. $z = 3x + 4y - 7$

13. $z - 5(x - 2) = 3(5 - y)$

14. $\pi(x - 1) = (1 - \pi)(y - z) + \pi$

15. Give a unit vector
  - **(a)** In the same direction as $\vec{v} = 2\vec{i} + 3\vec{j}$.
  - **(b)** Perpendicular to $\vec{v}$.

16. **(a)** Find a vector perpendicular to the plane $z = 2 + 3x - y$.
  - **(b)** Find a vector parallel to the plane.

17. **(a)** Find a vector perpendicular to the plane $z = 2x + 3y$.
  - **(b)** Find a vector parallel to the plane.

In Exercises 18–23, given vector $\vec{v} = 3\vec{i} + 4\vec{j}$ and force vector $\vec{F}$, find:

(a) The component of $\vec{F}$ parallel to $\vec{v}$.
(b) The component of $\vec{F}$ perpendicular to $\vec{v}$.
(c) The work, W, done by force $\vec{F}$ through displacement $\vec{v}$.

18. $\vec{F} = 4\vec{i} + \vec{j}$
19. $\vec{F} = 0.2\vec{i} - 0.5\vec{j}$
20. $\vec{F} = 9\vec{i} + 12\vec{j}$
21. $\vec{F} = -0.4\vec{i} + 0.3\vec{j}$
22. $\vec{F} = -3\vec{i} - 5\vec{j}$
23. $\vec{F} = -6\vec{i} - 8\vec{j}$

In Exercises 24–27, the force on an object is $\vec{F} = -20\vec{j}$. For vector $\vec{v}$, find:

(a) The component of $\vec{F}$ parallel to $\vec{v}$.
(b) The component of $\vec{F}$ perpendicular to $\vec{v}$.
(c) The work, W, done by force $\vec{F}$ through displacement $\vec{v}$.

24. $\vec{v} = 2\vec{i} + 3\vec{j}$
25. $\vec{v} = 5\vec{i} - \vec{j}$
26. $\vec{v} = 3\vec{j}$
27. $\vec{v} = 5\vec{i}$

Problems

28. Match the planes in (a)–(d) with **one or more** of the descriptions in (I)–(IV). No reasons needed.

(a) $3x - y + z = 0$
(b) $4x + y + 2z - 5 = 0$
(c) $x + y = 5$
(d) $x = 5$

I Goes through the origin.
II Has a normal vector parallel to the xy-plane.
III Goes through the point $(0, 5, 0)$.
IV Has a normal vector whose dot products with $\vec{i}$, $\vec{j}$, $\vec{k}$ are all positive.

29. Which pairs (if any) of vectors from the following list

(a) Are perpendicular?
(b) Are parallel?
(c) Have an angles less than $\pi/2$ between them?
(d) Have an angle of more than $\pi/2$ between them?

$$\vec{a} = \vec{i} - 3\vec{j} - \vec{k}, \quad \vec{b} = \vec{i} + \vec{j} + 2\vec{k},$$
$$\vec{c} = -2\vec{i} - \vec{j} + \vec{k}, \quad \vec{d} = -\vec{i} - \vec{j} + \vec{k}.$$

30. Which pairs of the vectors $\sqrt{3}\vec{i} + \vec{j}$, $3\vec{i} + \sqrt{3}\vec{j}$, $\vec{i} - \sqrt{3}\vec{j}$ are parallel and which are perpendicular?

31. Compute the angle between the vectors $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.

32. **(a)** Give a vector that is parallel to, but not equal to, $\vec{v} = 4\vec{i} + 3\vec{j}$.
(b) Give a vector that is perpendicular to $\vec{v}$.

33. What values of a make $\vec{v} = 2a\vec{i} - a\vec{j} + 16\vec{k}$ perpendicular to $\vec{w} = 5\vec{i} + a\vec{j} - \vec{k}$?

34. Let θ be the angle between $\vec{v}$ and $\vec{w}$, with $0 < \theta < \pi/2$. What is the effect on $\vec{v} \cdot \vec{w}$ of increasing each of the following quantities? Does $\vec{v} \cdot \vec{w}$ increase or decrease?

(a) $||\vec{v}\,||$
(b) θ

35. Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular, to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$.

36. Find angle BAC if $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$.

In Problems 37–42, find an equation of a plane that satisfies the given conditions.

37. Through $(1, 5, 2)$ perpendicular to $3\vec{i} - \vec{j} + 4\vec{k}$

38. Through $(2, -1, 3)$ perpendicular to $5\vec{i} + 4\vec{j} - \vec{k}$.

39. Perpendicular to the vector $2\vec{i} - 3\vec{j} + 7\vec{k}$ and passing through the point $(1, -1, 2)$.

40. Parallel to the plane $2x + 4y - 3z = 1$ and through the point $(1, 0, -1)$.

41. Through $(-2, 3, 2)$ and parallel to $3x + y + z = 4$.

42. Perpendicular to the vector $\vec{v} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and passing through the point $(4, 5, -2)$.

43. A plane has equation $z = 5x - 2y + 7$.

(a) Find a value of λ making the vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ normal to the plane.
(b) Find a value of a so that the point $(a + 1, a, a - 1)$ lies on the plane.

44. The points $(5, 0, 0)$, $(0, -3, 0)$, and $(0, 0, 2)$ form a triangle. Find the lengths of the sides of the triangle and each of its angles.

45. Let S be the triangle with vertices $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$.

(a) Find the length of the shortest side of S.
(b) Find the cosine of the angle BAC at vertex A.

46. A basketball gymnasium is 25 meters high, 80 meters wide and 200 meters long. For a half time stunt, the cheerleaders want to run two strings, one from each of the two corners above one basket to the diagonally opposite corners of the gym floor. What is the cosine of the angle made by the strings as they cross?

47. A 100-meter dash is run on a track in the direction of the vector $\vec{v} = 2\vec{i} + 6\vec{j}$. The wind velocity $\vec{w}$ is $5\vec{i} + \vec{j}$ km/hr. The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/hr. Will the race results be disqualified due to an illegal wind? Justify your answer.

48. An airplane is flying toward the southeast. Which of the following wind velocity vectors increases the plane's speed the most? Which slows down the plane the most?

$$\vec{w}_1 = -4\vec{i} - \vec{j} \qquad \vec{w}_2 = \vec{i} - 2\vec{j} \qquad \vec{w}_3 = -\vec{i} + 8\vec{j}$$

$$\vec{w}_4 = 10\vec{i} + 2\vec{j} \qquad \vec{w}_5 = 5\vec{i} - 2\vec{j}$$

49. A canoe is moving with velocity $\vec{v} = 5\vec{i} + 3\vec{j}$ m/sec relative to the water. The velocity of the current in the water is $\vec{c} = \vec{i} + 2\vec{j}$ m/sec.

(a) What is the speed of the current?
(b) What is the speed of the current in the direction of the canoe's motion?

50. A street vendor sells six items, with prices p_1 dollars per unit, p_2 dollars per unit, and so on. The vendor's price vector is $\vec{p} = (p_1, p_2, p_3, p_4, p_5, p_6) = (1.00, 3.50, 4.00, 2.75, 5.00, 3.00)$. The vendor sells q_1 units of the first item, q_2 units of the second item, and so on. The vendor's quantity vector is $\vec{q} = (q_1, q_2, q_3, q_4, q_5, q_6) = (43, 57, 12, 78, 20, 35)$. Find $\vec{p} \cdot \vec{q}$, give its units, and explain its significance to the vendor.

51. A course has four exams, weighted 10%, 15%, 25%, 50%, respectively. The class average on each of these exams is 75%, 91%, 84%, 87%, respectively. What do the vectors $\vec{a} = (0.75, 0.91, 0.84, 0.87)$ and $\vec{w} = (0.1, 0.15, 0.25, 0.5)$ represent, in terms of the course? Calculate the dot product $\vec{w} \cdot \vec{a}$. What does it represent, in terms of the course?

52. A consumption vector of three goods is defined by $\vec{x} = (x_1, x_2, x_3)$, where x_1, x_2 and x_3 are the quantities consumed of the three goods. A budget constraint is represented by the equation $\vec{p} \cdot \vec{x} = k$, where $\vec{p}$ is the price vector of the three goods and k is a constant. Show that the difference between two consumption vectors corresponding to points satisfying the same budget constraint is perpendicular to the price vector $\vec{p}$.

53. Find a vector that bisects the smaller of the two angles formed by $3\vec{i} + 4\vec{j}$ and $5\vec{i} - 12\vec{j}$.

54. What does Property 2 of the dot product in the box on page 702 say geometrically?

55. Show that the vectors $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ and $\vec{c}$ are perpendicular.

56. Show why each of the properties of the dot product in the box on page 702 follows from the algebraic definition of the dot product:

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

57. Show that if $\vec{u}$ and $\vec{v}$ are two vectors such that

$$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$$

for every vector $\vec{w}$, then

$$\vec{u} = \vec{v}.$$

58. Show that

$$\frac{\vec{u}}{\|\vec{u}\|^2} - \frac{\vec{v}}{\|\vec{v}\|^2} \text{ and } \frac{\vec{u}}{\|\vec{u}\|\|\vec{v}\|} - \frac{\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

have the same magnitude where $\vec{u}$ and $\vec{v}$ are nonzero vectors.

59. Figure 13.34 shows that, given three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, the sum of the components of $\vec{v}$ and $\vec{w}$ in the direction of $\vec{u}$ is the component of $\vec{v} + \vec{w}$ in the direction of $\vec{u}$. (Although the figure is drawn in two dimensions, this result is also true in three dimensions.) Use this figure to explain why the geometric definition of the dot product satisfies $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$.

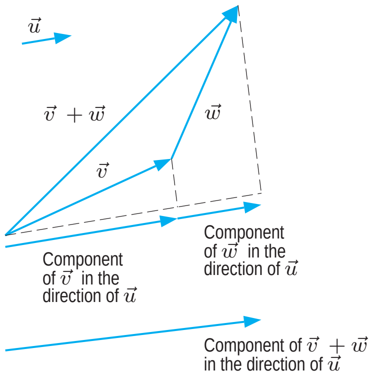

Figure 13.34: Component of $\vec{v} + \vec{w}$ in the direction of $\vec{u}$ is the sum of the components of $\vec{v}$ and $\vec{w}$ in that direction

60. **(a)** Using the geometric definition of the dot product, show that

$$\vec{u} \cdot (-\vec{v}) = -(\vec{u} \cdot \vec{v}).$$

[Hint: What happens to the angle when you multiply $\vec{v}$ by -1?]

(b) Using the geometric definition of the dot product, show that for any negative scalar λ

$$\vec{u} \cdot (\lambda\vec{v}) = \lambda(\vec{u} \cdot \vec{v})$$
$$(\lambda\vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v}).$$

61. The Law of Cosines for a triangle with side lengths a, b, and c, and with angle C opposite side c, says

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

On page 702, we used the Law of Cosines to show that the two definitions of the dot product are equivalent. In this problem, use the geometric definition of the dot product and its properties in the box on page 702 to prove the Law of Cosines. [Hint: Let $\vec{u}$ and $\vec{v}$ be the displacement vectors from C to the other two vertices, and express c^2 in terms of $\vec{u}$ and $\vec{v}$.]

62. Use the following steps and the results of Problems 59–60 to show (without trigonometry) that the geometric and algebraic definitions of the dot product are equivalent.

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be any vectors. Write $(\vec{u} \cdot \vec{v})_{\text{geom}}$ for the result of the dot product computed geometrically. Substitute $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and use Problems 59–60 to expand $(\vec{u} \cdot \vec{v})_{\text{geom}}$. Substitute for $\vec{v}$ and expand. Then calculate the dot products $\vec{i} \cdot \vec{i}$, $\vec{i} \cdot \vec{j}$, etc. geometrically.

63. For any vectors $\vec{v}$ and $\vec{w}$, consider the following function of t:

$$q(t) = (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w}).$$

(a) Explain why $q(t) \geq 0$ for all real t.
(b) Expand $q(t)$ as a quadratic polynomial in t using the properties on page 702.
(c) Using the discriminant of the quadratic, show that,

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

13.4 THE CROSS PRODUCT

In the previous section we combined two vectors to get a number, the dot product. In this section we see another way of combining two vectors, this time to get a vector, the *cross product*. Any two vectors in 3-space form a parallelogram. We define the cross product using this parallelogram.

The Area of a Parallelogram

Consider the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$ with an angle of θ between them. Then Figure 13.35 shows

$$\text{Area of parallelogram} = \text{Base} \cdot \text{Height} = \|\vec{v}\|\|\vec{w}\|\sin\theta.$$

How would we compute the area of the parallelogram if we were given $\vec{v}$ and $\vec{w}$ in components, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$? Project 1 on page 721 shows that if $\vec{v}$ and $\vec{w}$ are in the xy-plane, so $v_3 = w_3 = 0$, then

$$\text{Area of parallelogram} = |v_1w_2 - v_2w_1|.$$

What if $\vec{v}$ and $\vec{w}$ do not lie in the xy-plane? The cross product will enable us to compute the area of the parallelogram formed by any two vectors.

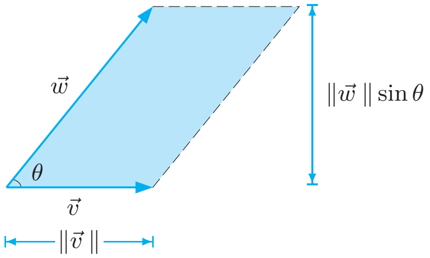

Figure 13.35: Parallelogram formed by $\vec{v}$ and $\vec{w}$ has Area $= \|\vec{v}\|\|\vec{w}\|\sin\theta$

Definition of the Cross Product

We define the cross product of the vectors $\vec{v}$ and $\vec{w}$, written $\vec{v} \times \vec{w}$, to be a vector perpendicular to both $\vec{v}$ and $\vec{w}$. The magnitude of this vector is the area of the parallelogram formed by the two vectors. The direction of $\vec{v} \times \vec{w}$ is given by the normal vector, $\vec{n}$, to the plane defined by $\vec{v}$ and $\vec{w}$. If we require that $\vec{n}$ be a unit vector, there are two choices for $\vec{n}$, pointing out of the plane in opposite directions. We pick one by the following rule (see Figure 13.36):

The right-hand rule: Place $\vec{v}$ and $\vec{w}$ so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles from $\vec{v}$ to $\vec{w}$; your thumb points in the direction of the normal vector, $\vec{n}$.