62. Use the following steps and the results of Problems 59–60 to show (without trigonometry) that the geometric and algebraic definitions of the dot product are equivalent.

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be any vectors. Write $(\vec{u} \cdot \vec{v})_{\text{geom}}$ for the result of the dot product computed geometrically. Substitute $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and use Problems 59–60 to expand $(\vec{u} \cdot \vec{v})_{\text{geom}}$. Substitute for $\vec{v}$ and expand. Then calculate the dot products $\vec{i} \cdot \vec{i}$, $\vec{i} \cdot \vec{j}$, etc. geometrically.

63. For any vectors $\vec{v}$ and $\vec{w}$, consider the following function of t:

$$q(t) = (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w}).$$

(a) Explain why $q(t) \geq 0$ for all real t.
(b) Expand $q(t)$ as a quadratic polynomial in t using the properties on page 702.
(c) Using the discriminant of the quadratic, show that,

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

13.4 THE CROSS PRODUCT

In the previous section we combined two vectors to get a number, the dot product. In this section we see another way of combining two vectors, this time to get a vector, the *cross product*. Any two vectors in 3-space form a parallelogram. We define the cross product using this parallelogram.

The Area of a Parallelogram

Consider the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$ with an angle of θ between them. Then Figure 13.35 shows

$$\text{Area of parallelogram} = \text{Base} \cdot \text{Height} = \|\vec{v}\|\|\vec{w}\|\sin\theta.$$

How would we compute the area of the parallelogram if we were given $\vec{v}$ and $\vec{w}$ in components, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$? Project 1 on page 721 shows that if $\vec{v}$ and $\vec{w}$ are in the xy-plane, so $v_3 = w_3 = 0$, then

$$\text{Area of parallelogram} = |v_1w_2 - v_2w_1|.$$

What if $\vec{v}$ and $\vec{w}$ do not lie in the xy-plane? The cross product will enable us to compute the area of the parallelogram formed by any two vectors.

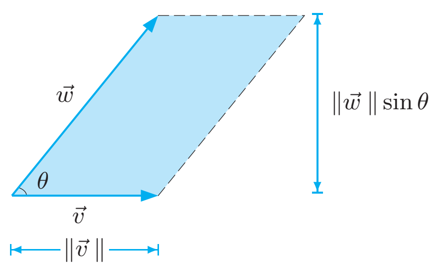

Figure 13.35: Parallelogram formed by $\vec{v}$ and $\vec{w}$ has Area $= \|\vec{v}\|\|\vec{w}\|\sin\theta$

Definition of the Cross Product

We define the cross product of the vectors $\vec{v}$ and $\vec{w}$, written $\vec{v} \times \vec{w}$, to be a vector perpendicular to both $\vec{v}$ and $\vec{w}$. The magnitude of this vector is the area of the parallelogram formed by the two vectors. The direction of $\vec{v} \times \vec{w}$ is given by the normal vector, $\vec{n}$, to the plane defined by $\vec{v}$ and $\vec{w}$. If we require that $\vec{n}$ be a unit vector, there are two choices for $\vec{n}$, pointing out of the plane in opposite directions. We pick one by the following rule (see Figure 13.36):

The right-hand rule: Place $\vec{v}$ and $\vec{w}$ so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles from $\vec{v}$ to $\vec{w}$; your thumb points in the direction of the normal vector, $\vec{n}$.

Like the dot product, there are two equivalent definitions of the cross product:

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

- **Geometric definition**
  If $\vec{v}$ and $\vec{w}$ are not parallel, then

$$\vec{v} \times \vec{w} = \begin{pmatrix}\text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w}\end{pmatrix} \vec{n} = (\|\vec{v}\|\|\vec{w}\| \sin\theta)\vec{n},$$

  where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.
- **Algebraic definition**

$$\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

  where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Notice that the magnitude of the $\vec{k}$ component is the area of a 2-dimensional parallelogram and the other components have a similar form. Problems 40 and 37 at the end of this section show that the geometric and algebraic definitions of the cross product give the same result.

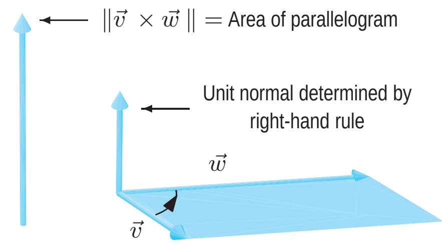

Figure 13.36: Area of parallelogram $= \|\vec{v} \times \vec{w}\|$

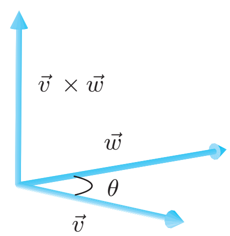

Figure 13.37: The cross product $\vec{v} \times \vec{w}$

Unlike the dot product, the cross product is only defined for three-dimensional vectors. The geometric definition shows us that the cross product is *rotation invariant*. Imagine the two vectors $\vec{v}$ and $\vec{w}$ as two metal rods welded together. Attach a third rod whose direction and length correspond to $\vec{v} \times \vec{w}$. (See Figure 13.37.) Then, no matter how we turn this set of rods, the third will still be the cross product of the first two.

The algebraic definition is more easily remembered by writing it as a 3×3 determinant. (See Appendix E.)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}.$$

Example 1 Find $\vec{i} \times \vec{j}$ and $\vec{j} \times \vec{i}$.

Solution The vectors $\vec{i}$ and $\vec{j}$ both have magnitude 1 and the angle between them is $\pi/2$. By the right-hand rule, the vector $\vec{i} \times \vec{j}$ is in the direction of $\vec{k}$, so $\vec{n} = \vec{k}$ and we have

$$\vec{i} \times \vec{j} = \left(\|\vec{i}\|\|\vec{j}\| \sin\frac{\pi}{2}\right)\vec{k} = \vec{k}.$$

Similarly, the right-hand rule says that the direction of $\vec{j} \times \vec{i}$ is $-\vec{k}$, so

$$\vec{j} \times \vec{i} = (\|\vec{j}\|\|\vec{i}\| \sin\frac{\pi}{2})(-\vec{k}) = -\vec{k}.$$

Similar calculations show that $\vec{j} \times \vec{k} = \vec{i}$ and $\vec{k} \times \vec{i} = \vec{j}$.

Example 2 For any vector $\vec{v}$, find $\vec{v} \times \vec{v}$.

Solution Since $\vec{v}$ is parallel to itself, $\vec{v} \times \vec{v} = \vec{0}$.

Example 3 Find the cross product of $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{w} = 3\vec{i} + \vec{k}$ and check that the cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

Solution Writing $\vec{v} \times \vec{w}$ as a determinant and expanding it into three two-by-two determinants, we have

$$\begin{aligned}\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \\ &= \vec{i}(1(1) - 0(-2)) - \vec{j}(2(1) - 3(-2)) + \vec{k}(2(0) - 3(1)) \\ &= \vec{i} - 8\vec{j} - 3\vec{k}.\end{aligned}$$

To check that $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$, we compute the dot product:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = (2\vec{i} + \vec{j} - 2\vec{k}) \cdot (\vec{i} - 8\vec{j} - 3\vec{k}) = 2 - 8 + 6 = 0.$$

Similarly,

$$\vec{w} \cdot (\vec{v} \times \vec{w}) = (3\vec{i} + 0\vec{j} + \vec{k}) \cdot (\vec{i} - 8\vec{j} - 3\vec{k}) = 3 + 0 - 3 = 0.$$

Thus, $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

Properties of the Cross Product

The right-hand rule tells us that $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ point in opposite directions. The magnitudes of $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are the same, so $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$. (See Figure 13.38.)

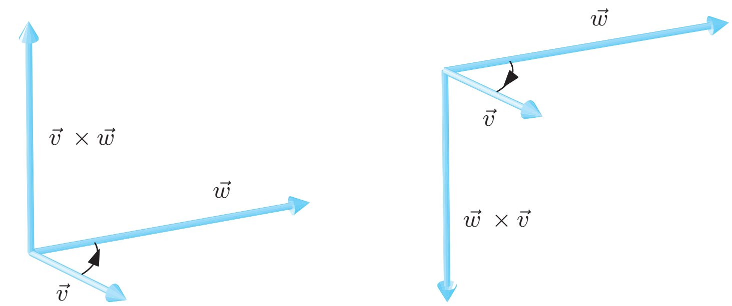

Figure 13.38: Diagram showing $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

This explains the first of the following properties. The other two are derived in Problems 31, 32, and 40 at the end of this section.

Properties of the Cross Product

For vectors $\vec{u}, \vec{v}, \vec{w}$ and scalar λ

1. $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$
2. $(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda\vec{w})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

The Equivalence of the Two Definitions of the Cross Product

Problem 40 on page 716 uses geometric arguments to show that the cross product distributes over addition. Problem 37 then shows how the formula in the algebraic definition of the cross product can be derived from the geometric definition.

The Equation of a Plane Through Three Points

The equation of a plane is determined by a point $P_0 = (x_0, y_0, z_0)$ on the plane, and a normal vector, $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

However, a plane can also be determined by three points on it (provided they do not lie on a line). In that case we can find an equation of the plane by first determining two vectors in the plane and then finding a normal vector using the cross product, as in the following example.

Example 4 Find an equation of the plane containing the points $P = (1, 3, 0)$, $Q = (3, 4, -3)$, and $R = (3, 6, 2)$.

Solution Since the points P and Q are in the plane, the displacement vector between them, $\overrightarrow{PQ}$, is in the plane, where

$$\overrightarrow{PQ} = (3 - 1)\vec{i} + (4 - 3)\vec{j} + (-3 - 0)\vec{k} = 2\vec{i} + \vec{j} - 3\vec{k}.$$

The displacement vector $\overrightarrow{PR}$ is also in the plane, where

$$\overrightarrow{PR} = (3 - 1)\vec{i} + (6 - 3)\vec{j} + (2 - 0)\vec{k} = 2\vec{i} + 3\vec{j} + 2\vec{k}.$$

Thus, a normal vector, $\vec{n}$, to the plane is given by

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix} = 11\vec{i} - 10\vec{j} + 4\vec{k}.$$

Since the point $(1, 3, 0)$ is on the plane, the equation of the plane is

$$11(x - 1) - 10(y - 3) + 4(z - 0) = 0,$$

which simplifies to

$$11x - 10y + 4z = -19.$$

You should check that P, Q, and R satisfy the equation of the plane.

Areas and Volumes Using the Cross Product and Determinants

We can use the cross product to calculate the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$. We say that $\vec{v} \times \vec{w}$ is the *area vector* of the parallelogram. The geometric definition of the cross product tells us that $\vec{v} \times \vec{w}$ is normal to the parallelogram and gives us the following result:

Area of a parallelogram with edges $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ is given by

$$\text{Area} = \|\vec{v} \times \vec{w}\|, \qquad \text{where} \quad \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example 5 Find the area of the parallelogram with edges $\vec{v} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{w} = \vec{i} + 3\vec{j} + 2\vec{k}$.

Solution We calculate the cross product:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = (2+9)\vec{i} - (4+3)\vec{j} + (6-1)\vec{k} = 11\vec{i} - 7\vec{j} + 5\vec{k}.$$

The area of the parallelogram with edges $\vec{v}$ and $\vec{w}$ is the magnitude of the vector $\vec{v} \times \vec{w}$:

$$\text{Area} = \|\vec{v} \times \vec{w}\| = \sqrt{11^2 + (-7)^2 + 5^2} = \sqrt{195}.$$

Volume of a Parallelepiped

Consider the parallelepiped with sides formed by $\vec{a}$, $\vec{b}$, and $\vec{c}$. (See Figure 13.39.) Since the base is formed by the vectors $\vec{b}$ and $\vec{c}$, we have

$$\text{Area of base of parallelepiped} = \|\vec{b} \times \vec{c}\|.$$

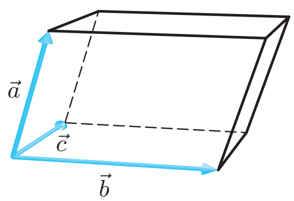

Figure 13.39: Volume of a Parallelepiped

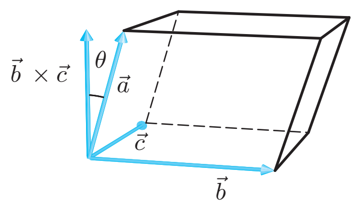

Figure 13.40: The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are called a right-handed set

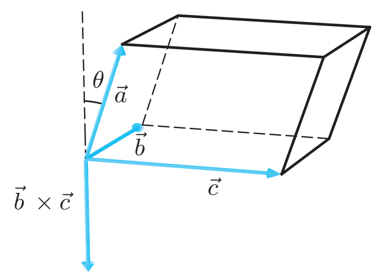

Figure 13.41: The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are called a left-handed set

The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be arranged either as in Figure 13.40 or as in Figure 13.41. In either case,

$$\text{Height of parallelepiped} = \|\vec{a}\| \cos\theta,$$

where θ is the angle shown in the figures. In Figure 13.40 the angle θ is less than $\pi/2$, so the product, $(\vec{b} \times \vec{c}) \cdot \vec{a}$, called the *triple product*, is positive. Thus, in this case

$$\text{Volume of parallelepiped} = \text{Base} \cdot \text{Height} = \|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos\theta = (\vec{b} \times \vec{c}) \cdot \vec{a}.$$

In Figure 13.41, the angle, $\pi - \theta$, between $\vec{a}$ and $\vec{b} \times \vec{c}$ is more than $\pi/2$, so the product $(\vec{b} \times \vec{c}) \cdot \vec{a}$ is negative. Thus, in this case we have

$$\begin{aligned} \text{Volume} = \text{Base} \cdot \text{Height} &= \|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos\theta = -\|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos(\pi - \theta) \\ &= -(\vec{b} \times \vec{c}) \cdot \vec{a} = \left|(\vec{b} \times \vec{c}) \cdot \vec{a}\right|. \end{aligned}$$

Therefore, in both cases the volume is given by $\left|(\vec{b} \times \vec{c}) \cdot \vec{a}\right|$. Using determinants, we can write

Volume of a parallelepiped with edges $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

$$\text{Volume} = \left|(\vec{b} \times \vec{c}) \cdot \vec{a}\right| = \text{Absolute value of the determinant} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Exercises and Problems for Section 13.4

Exercises

In Exercises 1–7, use the algebraic definition to find $\vec{v}\times\vec{w}$.

1. $\vec{v}=\vec{k},\vec{w}=\vec{j}$
2. $\vec{v}=-\vec{i},\vec{w}=\vec{j}+\vec{k}$
3. $\vec{v}=\vec{i}+\vec{k},\vec{w}=\vec{i}+\vec{j}$
4. $\vec{v}=\vec{i}+\vec{j}+\vec{k},\vec{w}=\vec{i}+\vec{j}+-\vec{k}$
5. $\vec{v}=2\vec{i}-3\vec{j}+\vec{k},\vec{w}=\vec{i}+2\vec{j}-\vec{k}$
6. $\vec{v}=2\vec{i}-\vec{j}-\vec{k},\vec{w}=-6\vec{i}+3\vec{j}+3\vec{k}$
7. $\vec{v}=-3\vec{i}+5\vec{j}+4\vec{k},\vec{w}=\vec{i}-3\vec{j}-\vec{k}$

Use the geometric definition in Exercises 8–9 to find:

8. $2\vec{i}\times(\vec{i}+\vec{j})$

9. $(\vec{i}+\vec{j})\times(\vec{i}-\vec{j})$

In Exercises 10–11, use the properties on page 712 to find:

10. $\left((\vec{i}+\vec{j})\times\vec{i}\right)\times\vec{j}$

11. $(\vec{i}+\vec{j})\times(\vec{i}\times\vec{j})$

Find an equation for the plane through the points in Exercises 12–13.

12. $(1,0,0),(0,1,0),(0,0,1)$.

13. $(3,4,2),(-2,1,0),(0,2,1)$.

14. For $\vec{a}=3\vec{i}+\vec{j}-\vec{k}$ and $\vec{b}=\vec{i}-4\vec{j}+2\vec{k}$, find $\vec{a}\times\vec{b}$ and check that it is perpendicular to both $\vec{a}$ and $\vec{b}$.

15. If $\vec{v}=3\vec{i}-2\vec{j}+4\vec{k}$ and $\vec{w}=\vec{i}+2\vec{j}-\vec{k}$, find $\vec{v}\times\vec{w}$ and $\vec{w}\times\vec{v}$. What is the relation between the two answers?

Problems

16. Find a vector parallel to the line of intersection of the planes given by the equations $2x-3y+5z=2$ and $4x+y-3z=7$.

17. Find the equation of the plane through the origin which is perpendicular to the line of intersection of the planes in Problem 16.

18. Find the equation of the plane through the point $(4,5,6)$ and perpendicular to the line of intersection of the planes in Problem 16.

19. Find an equation for the plane through the origin containing the points $(1,3,0)$ and $(2,4,1)$.

20. Find a vector parallel to the line of intersection of the two planes $4x-3y+2z=12$ and $x+5y-z=25$.

21. Find a vector parallel to the intersection of the planes $2x-3y+5z=2$ and $4x+y-3z=7$.

22. Find the equation of the plane through the origin which is perpendicular to the line of intersection of the planes in Problem 21.

23. Find the equation of the plane through the point $(4,5,6)$ which is perpendicular to the line of intersection of the planes in Problem 21.

24. Find the equation of a plane through the origin and perpendicular to $x-y+z=5$ and $2x+y-2z=7$.

25. Let $P=(0,1,0),Q=(-1,1,2),R=(2,1,-1)$. Find

- **(a)** The area of the triangle PQR.
- **(b)** The equation for a plane that contains P, Q, and R.

26. Let $A=(-1,3,0)$, $B=(3,2,4)$, and $C=(1,-1,5)$.

- **(a)** Find an equation for the plane that passes through these three points.
- **(b)** Find the area of the triangle determined by these three points.

27. If $\vec{v}$ and $\vec{w}$ are both parallel to the xy-plane, what can you conclude about $\vec{v}\times\vec{w}$? Explain.

28. Suppose $\vec{v}\cdot\vec{w}=5$ and $||\vec{v}\times\vec{w}||=3$, and the angle between $\vec{v}$ and $\vec{w}$ is θ. Find

- **(a)** $\tan\theta$
- **(b)** θ.

29. If $\vec{v}\times\vec{w}=2\vec{i}-3\vec{j}+5\vec{k}$, and $\vec{v}\cdot\vec{w}=3$, find $\tan\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$.

30. The point P in Figure 13.42 has position vector $\vec{v}$ obtained by rotating the position vector $\vec{r}$ of the point (x,y) by 90° counterclockwise about the origin.

- **(a)** Use the geometric definition of the cross product to explain why $\vec{v}=\vec{k}\times\vec{r}$.
- **(b)** Find the coordinates of P.

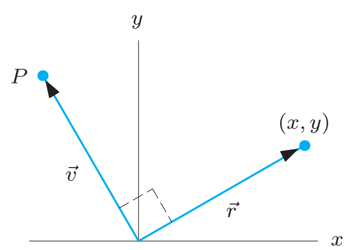

Figure 13.42

31. Use the algebraic definition to check that

$$\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}).$$

32. If $\vec{v}$ and $\vec{w}$ are nonzero vectors, use the geometric definition of the cross product to explain why

$$(\lambda\vec{v})\times\vec{w}=\lambda(\vec{v}\times\vec{w})=\vec{v}\times(\lambda\vec{w}).$$

Consider the cases $\lambda>0$, and $\lambda=0$, and $\lambda<0$ separately.

33. Use a parallelepiped to show that $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

34. Show that $\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$.

35. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, show that

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.$$

Geometrically, what does this imply about $\vec{a}$, $\vec{b}$, and $\vec{c}$?

36. If $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ and $\vec{c} = c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$ are any three vectors in space, show that

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

37. Use the fact that $\vec{i}\times\vec{i}=\vec{0}$, $\vec{i}\times\vec{j}=\vec{k}$, $\vec{i}\times\vec{k}=-\vec{j}$, and so on, together with the properties on page 712 to derive the algebraic definition for the cross product.

38. In this problem, we arrive at the algebraic definition for the cross product by a different route. Let $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$. We seek a vector $\vec{v} = x\vec{i}+y\vec{j}+z\vec{k}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$. Use this requirement to construct two equations for x, y, and z. Eliminate x and solve for y in terms of z. Then eliminate y and solve for x in terms of z. Since z can be any value whatsoever (the direction of $\vec{v}$ is unaffected), select the value for z which eliminates the denominator in the equation you obtained. How does the resulting expression for $\vec{v}$ compare to the formula we derived on page 711?

39. For vectors $\vec{a}$ and $\vec{b}$, let $\vec{c} = \vec{a}\times(\vec{b}\times\vec{a})$.

(a) Show that $\vec{c}$ lies in the plane containing $\vec{a}$ and $\vec{b}$.
(b) Use Problems 33 and 34 to show that $\vec{a}\cdot\vec{c}=0$ and $\vec{b}\cdot\vec{c} = \|\vec{a}\|^2\|\vec{b}\|^2-(\vec{a}\cdot\vec{b})^2$.
(c) Show that

$$\vec{a}\times(\vec{b}\times\vec{a}) = \|\vec{a}\|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}.$$

40. Use the result of Problem 33 to show that the cross product distributes over addition. First, use distributivity for the dot product to show that for any vector $\vec{d}$,

$$[(\vec{a}+\vec{b})\times\vec{c}]\cdot\vec{d} = [(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})]\cdot\vec{d}.$$

Next, show that for any vector $\vec{d}$,

$$[((\vec{a}+\vec{b})\times\vec{c})-(\vec{a}\times\vec{c})-(\vec{b}\times\vec{c})]\cdot\vec{d}=0.$$

Finally, explain why you can conclude that

$$(\vec{a}+\vec{b})\times\vec{c} = (\vec{a}\times\vec{c})+(\vec{b}\times\vec{c}).$$

41. Figure 13.43 shows the tetrahedron determined by three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. The *area vector* of a face is a vector perpendicular to the face, pointing outward, whose magnitude is the area of the face. Show that the sum of the four outward pointing area vectors of the faces equals the zero vector.

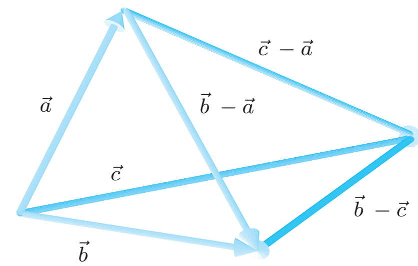

Figure 13.43

In Problems 42–44, find the vector representing the area of a surface. The magnitude of the vector equals the magnitude of the area; the direction is perpendicular to the surface. Since there are two perpendicular directions, we pick one by giving an orientation for the surface.

42. The rectangle with vertices $(0,0,0)$, $(0,1,0)$, $(2,1,0)$, and $(2,0,0)$, oriented so that it faces downward.

43. The circle of radius 2 in the yz-plane, facing in the direction of the positive x-axis.

44. The triangle ABC, oriented upward, where $A = (1,2,3)$, $B=(3,1,2)$, and $C=(2,1,3)$.

45. This problem relates the area of a parallelogram S lying in the plane $z = mx+ny+c$ to the area of its projection R in the xy-plane. Let S be determined by the vectors $\vec{u} = u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$ and $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$. See Figure 13.44.

(a) Find the area of S
(b) Find the area of R
(c) Find m and n in terms of the components of $\vec{u}$ and $\vec{v}$.
(d) Show that

$$\text{Area of } S = \sqrt{1+m^2+n^2}\cdot \text{ Area of } R$$

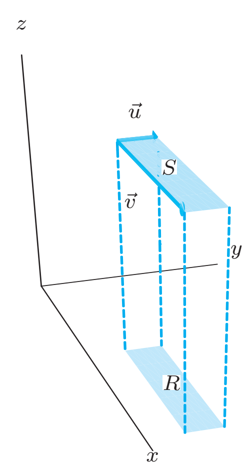

Figure 13.44

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Vectors**
  Geometric definition of vector addition, subtraction and scalar multiplication, resolving into $\vec{i}$, $\vec{j}$, and $\vec{k}$ components, magnitude of a vector, algebraic properties of addition and scalar multiplication.
- **Dot Product**
  Geometric and algebraic definition, algebraic properties, using dot products to find angles and determine perpendicularity, the equation of a plane with given normal vector passing through a given point, projection of a vector in a direction given by a unit vector.
- **Cross Product**
  Geometric and algebraic definition, algebraic properties, cross product and volume, finding the equation of a plane through three points.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER THIRTEEN

Exercises

In Exercises 1–2, is the quantity a vector or a scalar? Compute it.

1. $\vec{u}\cdot\vec{v}$, where $\vec{u} = 2\vec{i} - 3\vec{j} - 4\vec{k}$ and $\vec{v} = \vec{k} - \vec{j}$

2. $\vec{u}\times\vec{v}$, where $\vec{u} = 2\vec{i} - 3\vec{j} - 4\vec{k}$ and $\vec{v} = 3\vec{i} - \vec{j} + \vec{k}$.

3. Resolve the vectors in Figure 13.45 into components.

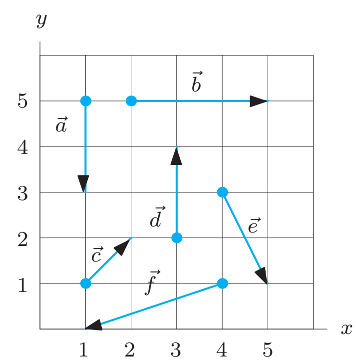

Figure 13.45

4. Resolve vector $\vec{v}$ into components if $\|\vec{v}\| = 8$ and the direction of $\vec{v}$ is shown in Figure 13.46.

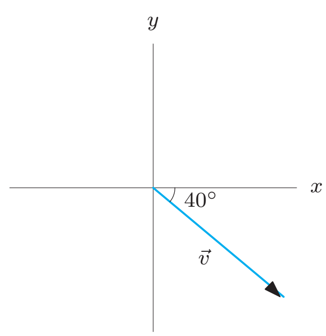

Figure 13.46

For Exercises 5–7, perform the indicated operations on the following vectors:

$$\vec{c} = \vec{i} + 6\vec{j}, \quad \vec{x} = -2\vec{i} + 9\vec{j}, \quad \vec{y} = 4\vec{i} - 7\vec{j}.$$

5. $5\vec{c}$ **6.** $\vec{c} + \vec{x} + \vec{y}$ **7.** $\|\vec{x} - \vec{c}\|$

In Exercises 8–17, use $\vec{v} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{w} = \vec{i} - \vec{j} + 2\vec{k}$ to calculate the given quantities.

8. $\vec{v} + 2\vec{w}$

9. $3\vec{v} - \vec{w} - \vec{v}$

10. $\|\vec{v} + \vec{w}\|$

11. $\vec{v}\cdot\vec{w}$

12. $\vec{v}\times\vec{w}$

13. $\vec{v}\times\vec{v}$

14. $(\vec{v}\cdot\vec{w})\vec{v}$

15. $(\vec{v}\times\vec{w})\cdot\vec{w}$

16. $(\vec{v}\times\vec{w})\times\vec{w}$

17. $(\vec{v}\times\vec{w})\times(\vec{v}\times\vec{w})$

In Exercises 18–19, find a normal vector to the plane.

18. $2x + y - z = 5$

19. $2(x - z) = 3(x + y)$

20. Find the equation of the plane through the origin which is parallel to $z = 4x - 3y + 8$.

21. Let $\vec{v} = 3\vec{i} + 2\vec{j} - 2\vec{k}$ and $\vec{w} = 4\vec{i} - 3\vec{j} + \vec{k}$. Find each of the following:

(a) $\vec{v}\cdot\vec{w}$
(b) $\vec{v}\times\vec{w}$
(c) A vector of length 5 parallel to vector $\vec{v}$
(d) The angle between vectors $\vec{v}$ and $\vec{w}$
(e) The component of $\vec{v}$ in the direction of $\vec{w}$
(f) A vector perpendicular to vector $\vec{v}$
(g) A vector perpendicular to both vectors $\vec{v}$ and $\vec{w}$