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Hughes-Hallett, Calculus: Single and Multivariable, 4/e

Calculus I, II, and Vector Calculus

Chapter 7. Integration Go

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7.1 Integration by Substitution

In Chapter 3, we learned rules to differentiate any function obtained by combining constants, powers of x, sin x, cos x, e^x, and ln x, using addition, multiplication, division, or composition of functions. Such functions are called *elementary*.

In the next few sections, we introduce two methods of antidifferentiation: substitution and integration by parts, which reverse the chain and product rules, respectively. However, there is a great difference between looking for derivatives and looking for antiderivatives. Every elementary function has elementary derivatives, but many elementary functions do not have elementary antiderivatives. Some examples are $\sqrt{x^3+1}$, $(\sin x)/x$, and e^{-x^2}. These are not exotic functions, but ordinary functions that arise naturally.

The Guess-and-Check Method

A good strategy for finding simple antiderivatives is to *guess* an answer (using knowledge of differentiation rules) and then *check* the answer by differentiating it. If we get the expected result, then we're done; otherwise, we revise the guess and check again.

The method of guess-and-check is useful in reversing the chain rule. According to the chain rule,

$$\frac{d}{dx}(f(g(x))) = \underbrace{f'}_{\text{Derivative of outside}}\overbrace{(g(x))}^{\text{Inside}} \cdot \underbrace{g'(x)}_{\text{Derivative of inside}}.$$

Thus, any function which is the result of applying the chain rule is the product of two factors: the "derivative of the outside" and the "derivative of the inside." If a function has this form, its antiderivative is $f(g(x))$.

Example 1

Find $\int 3x^2\cos(x^3)\,dx$.

Solution

The function $3x^2\cos(x^3)$ looks like the result of applying the chain rule: there is an "inside" function x^3 and its derivative $3x^2$ appears as a factor. Since the outside function is a cosine which has a sine as an antiderivative, we guess $\sin(x^3)$ for the antiderivative. Differentiating to check gives

$$\frac{d}{dx}(\sin(x^3)) = \cos(x^3)\cdot(3x^2)\ .$$

Since this is what we began with, we know that

$$\int 3x^2\cos(x^3)\,dx = \sin(x^3) + C\ .$$

The basic idea of this method is to try to find an inside function whose derivative appears as a factor. This works even when the derivative is missing a constant factor, as in the next example.

Example 2

Find $\int te^{(t^2+1)}dt$.

Solution

It looks like t^2+1 is an inside function. So we guess $e^{(t^2+1)}$ for the antiderivative, since taking the derivative of an exponential results in the reappearance of the exponential together with other terms from the chain rule. Now we check:

$$\frac{d}{dt}\left(e^{(t^2+1)}\right) = \left(e^{(t^2+1)}\right)\cdot 2t\ .$$

The original guess was too large by a factor of 2. We change the guess to $\frac{1}{2}e^{(t^2+1)}$ and check again:

$$\frac{d}{dt}\left(\frac{1}{2}e^{(t^2+1)}\right) = \frac{1}{2}e^{(t^2+1)}\cdot 2t = e^{(t^2+1)}\cdot t\ .$$

Thus, we know that

$$\int te^{(t^2+1)}dt = \frac{1}{2}e^{(t^2+1)} + C\,.$$

Example 3

Find $\int x^3\sqrt{x^4+5}\,dx$

Solution

Here the inside function is x^4+5, and its derivative appears as a factor, with the exception of a missing 4. Thus, the integrand we have is more or less of the form

$$g'(x)\sqrt{g(x)},$$

with $g(x) = x^4+5$. Since $x^{3/2}/(3/2)$ is an antiderivative of the outside function $\sqrt{x}$, we might guess that an antiderivative is

$$\frac{(g(x))^{3/2}}{3/2} = \frac{(x^4+5)^{3/2}}{3/2}\,.$$

Let's check and see:

$$\frac{d}{dx}\left(\frac{(x^4+5)^{3/2}}{3/2}\right) = \frac{3}{2}\,\frac{(x^4+5)^{1/2}}{3/2}\cdot 4x^3 = 4x^3(x^4+5)^{1/2},$$

so $\dfrac{(x^4+5)^{3/2}}{3/2}$ is too big by a factor of 4. The correct antiderivative is

$$\frac{1}{4}\,\frac{(x^4+5)^{3/2}}{3/2} = \frac{1}{6}(x^4+5)^{3/2}\,.$$

Thus

$$\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C\,.$$

As a final check:

$$\frac{d}{dx}\left(\frac{1}{6}(x^4+5)^{3/2}\right) = \frac{1}{6}\cdot\frac{3}{2}(x^4+5)^{1/2}\cdot 4x^3 = x^3(x^4+5)^{1/2}\,.$$

As we have seen in the preceding examples, antidifferentiating a function often involves “correcting for” constant factors: if differentiation produces an extra factor of 4, antidifferentiation will require a factor of $\frac{1}{4}$.

The Method of Substitution

When the integrand is complicated, it helps to formalize this guess-and-check method as follows:

To Make a Substitution

Let *w* be the “inside function” and $dw = w'(x)\,dx = \frac{dw}{dx}dx$.

Let's redo the first example using a substitution.

Example 4

Find $\int 3x^2\cos(x^3)\,dx$.

Solution

As before, we look for an inside function whose derivative appears—in this case x^3. We let $w = x^3$. Then $dw = w'(x)\,dx = 3x^2dx$. The original integrand can now be completely rewritten in terms of the new variable *w*:

$$\int 3x^2\cos(x^3)\,dx = \int \cos\underbrace{(x^3)}_{w} \cdot \underbrace{3x^2dx}_{dw} = \int \cos w\,dw = \sin w + C = \sin(x^3) + C\,.$$

By changing the variable to *w*, we can simplify the integrand. We now have $\cos w$, which can be antidifferentiated more easily. The final step, after antidifferentiating, is to convert back to the original variable, *x*.

Why Does Substitution Work?

The substitution method makes it look as if we can treat dw and dx as separate entities, even canceling them in the equation $dw = (dw\,/\,dx)dx$. Let's see why this works. Suppose we have an integral of the form $\int f(g(x))g'(x)\,dx$, where $g(x)$ is the inside function and $f(x)$ is the outside function. If *F* is an antiderivative of *f*, then $F' = f$, and by the chain rule $\frac{d}{dx}(F(g(x))) = f(g(x))g'(x)$. Therefore,

$$\int f(g(x))g'(x)dx = F(g(x)) + C\,.$$

Now write $w = g(x)$ and $dw\,/\,dx = g'(x)$ on both sides of this equation:

$$\int f(w)\frac{dw}{dx}dx = F(w) + C\,.$$

On the other hand, knowing that $F' = f$ tells us that

$$\int f(w)\ dw = F(w) + C\,.$$

Thus, the following two integrals are equal:

$$\int f(w)\frac{dw}{dx}\ dx = \int f(w)\ dw\,.$$

Substituting *w* for the inside function and writing $dw = w'(x)\ dx$ leaves the indefinite integral unchanged.

Let's revisit the second example that we did by guess-and-check.

Example 5

Find $\int te^{(t^2+1)}dt$.

Solution

Here the inside function is $t^2 + 1$, with derivative $2t$. Since there is a factor of *t* in the integrand, we try

$$w = t^2 + 1\,.$$

Then

$$dw = w'(t)\ dt = 2t\ dt\,.$$

Notice, however, that the original integrand has only $t\ dt$, not $2t\ dt$. We therefore write

$$\frac{1}{2}dw = t\ dt$$

and then substitute:

$$\int te^{(t^2+1)}dt = \int e^{\overbrace{(t^2+1)}^{w}} \cdot \underbrace{t\,dt}_{\frac{1}{2}dw} = \int e^w \frac{1}{2}dw = \frac{1}{2}\int e^w dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{(t^2+1)} + C\,.$$

This gives the same answer as we found using guess-and-check.

Why didn't we put $\frac{1}{2}\int e^w dw = \frac{1}{2}e^w + \frac{1}{2}C$ in the preceding example? Since the constant C is arbitrary, it doesn't really matter whether we add C or $\frac{1}{2}C$. The convention is always to add C to whatever antiderivative we have calculated.

Now let's redo the third example that we solved previously by guess-and-check.

Example 6

Find $\int x^3\sqrt{x^4+5}\,dx$.

Solution

The inside function is x^4+5, with derivative $4x^3$. The integrand has a factor of x^3, and since the only thing missing is a constant factor, we try

$$w = x^4 + 5\,.$$

Then

$$dw = w'(x)\,dx = 4x^3dx,$$

giving

$$\frac{1}{4}dw = x^3dx\,.$$

Thus,

$$\int x^3\sqrt{x^4+5}\,dx = \int \sqrt{w}\frac{1}{4}dw = \frac{1}{4}\int w^{1/2}dw = \frac{1}{4}\cdot\frac{w^{3/2}}{3/2} + C = \frac{1}{6}(x^4+5)^{3/2} + C\,.$$

Once again, we get the same result as with guess-and-check.

Warning

We saw in the preceding examples that we can apply the substitution method when a **constant** factor is missing from the derivative of the inside function. However, we may not be able to use substitution if anything other than a constant factor is missing. For example, setting $w = x^4 + 5$ to find

$$\int x^2 \sqrt{x^4 + 5}\, dx$$

does us no good because $x^2 dx$ is not a constant multiple of $dw = 4x^3 dx$. Substitution works if the integrand contains the derivative of the inside function, *to within a constant factor*.

Some people prefer the substitution method over guess-and-check since it is more systematic, but both methods achieve the same result. For simple problems, guess-and-check can be faster.

Example 7

Find $\int e^{\cos\theta} \sin\theta\, d\theta$.

Solution

We let $w = \cos\theta$ since its derivative is $-\sin\theta$ and there is a factor of $\sin\theta$ in the integrand. This gives

$$dw = w'(\theta)\, d\theta = -\sin\theta\, d\theta,$$

so

$$-dw = \sin\theta\, d\theta\,.$$

Thus

$$\int e^{\cos\theta} \sin\theta\, d\theta = \int e^w (-dw) = (-1)\int e^w dw = -e^w + C = -e^{\cos\theta} + C\,.$$

Example 8

Find $\displaystyle\int \frac{e^t}{1+e^t}\,dt$.

Solution

Observing that the derivative of $1+e^t$ is e^t, we see $w = 1+e^t$ is a good choice. Then $dw = e^t dt$, so that

$$\begin{aligned}\int \frac{e^t}{1+e^t}dt = \int \frac{1}{1+e^t}e^t dt = \int \frac{1}{w}dw &= \ln|w| + C \\ &= \ln\left|1+e^t\right| + C \\ &= \ln(1+e^t) + C \qquad (\text{Since } (1+e^t) \text{ is always positive.})\end{aligned}$$

Since the numerator is $e^t dt$, we might also have tried $w = e^t$. This substitution leads to the integral $\displaystyle\int (1\,/\,(1+w))dw$, which is better than the original integral but requires another substitution, $u = 1+w$, to finish. There are often several different ways of doing an integral by substitution.

Notice the pattern in the previous example: having a function in the denominator and its derivative in the numerator leads to a natural logarithm. The next example follows the same pattern.

Example 9

Find $\displaystyle\int \tan\theta\,d\theta$.

Solution

Recall that $\tan\theta = (\sin\theta)\,/\,(\cos\theta)$. If $w = \cos\theta$, then $dw = -\sin\theta\,d\theta$, so

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}d\theta = \int \frac{-dw}{w} = -\ln|w| + C = -\ln|\cos\theta| + C\,.$$

Definite Integrals by Substitution

Example 10

Compute $\int_0^2 xe^{x^2}dx$

Solution

To evaluate this definite integral using the Fundamental Theorem of Calculus, we first need to find an antiderivative of $f(x) = xe^{x^2}$. The inside function is x^2, so we let $w = x^2$. Then $dw = 2x\,dx$, so $\frac{1}{2}dw = x\,dx$. Thus,

$$\int xe^{x^2}dx = \int e^w \frac{1}{2}dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{x^2} + C\,.$$

Now we find the definite integral

$$\int_0^2 xe^{x^2}dx = \frac{1}{2}e^{x^2}\Big|_0^2 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)\,.$$

There is another way to look at the same problem. After we established that

$$\int xe^{x^2}dx = \frac{1}{2}e^w + C,$$

our next two steps were to replace *w* by x^2, and then *x* by 2 and 0. We could have directly replaced the original limits of integration, $x = 0$ and $x = 2$, by the corresponding *w* limits. Since $w = x^2$, the *w* limits are $w = 0^2 = 0$ (when $x = 0$) and $w = 2^2 = 4$ (when $x = 2$), so we get

$$\int_{x=0}^{x=2} xe^{x^2}dx = \frac{1}{2}\int_{w=0}^{w=4} e^w dw = \frac{1}{2}e^w\Big|_0^4 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)\,.$$

As we would expect, both methods give the same answer.

To Use Substitution to Find Definite Integrals

Either

- Compute the indefinite integral, expressing an antiderivative in terms of the original variable, and then evaluate the result at the original limits,

  or

- Convert the original limits to new limits in terms of the new variable and do not convert the antiderivative back to the original variable.

Example 11

Evaluate $\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta} d\theta$.

Solution

To use substitution, we must decide what *w* should be. There are two possible inside functions, $\tan\theta$ and $\cos\theta$. Now

$$\frac{d}{d\theta}(\tan\theta) = \frac{1}{\cos^2\theta} \quad \text{and} \quad \frac{d}{d\theta}(\cos\theta) = -\sin\theta,$$

and since the integral contains a factor of $1/\cos^2\theta$ but not of $\sin\theta$, we try $w = \tan\theta$. Then $dw = (1/\cos^2\theta)\,d\theta$. When $\theta = 0$, $w = \tan 0 = 0$, and when $\theta = \pi/4$, $w = \tan(\pi/4) = 1$, so

$$\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} (\tan\theta)^3 \cdot \frac{1}{\cos^2\theta} d\theta = \int_0^1 w^3 dw = \frac{1}{4}w^4\Big|_0^1 = \frac{1}{4}.$$

Example 12

Evaluate $\int_1^3 \frac{dx}{5-x}$.

Solution

Let $w = 5 - x$, so $dw = -dx$. When $x = 1$, $w = 4$, and when $x = 3$, $w = 2$, so

$$\int_1^3 \frac{dx}{5-x} = \int_4^2 \frac{-dw}{w} = -\ln|w|\Big|_4^2 = -(\ln 2 - \ln 4) = \ln\left(\frac{4}{2}\right) = \ln 2 \approx 0.69 .$$

Notice that we write the limit $w = 4$ at the bottom, even though it is larger than $w = 2$, because $w = 4$ corresponds to the lower limit $x = 1$.

More Complex Substitutions

In the examples of substitution presented so far, we guessed an expression for *w* and hoped to find dw (or some constant multiple of it) in the integrand. What if we are not so lucky? It turns out that it often works to let *w* be some messy expression contained inside, say, a cosine or under a root, even if we cannot see immediately how such a substitution helps.

Example 13

Find $\int \sqrt{1+\sqrt{x}}\, dx$.

Solution

This time, the derivative of the inside function is nowhere to be seen. Nevertheless, we try $w = 1 + \sqrt{x}$. Then $w - 1 = \sqrt{x}$, so $(w-1)^2 = x$. Therefore $2(w-1)dw = dx$. We have

$$\begin{aligned}\int \sqrt{1+\sqrt{x}}\, dx &= \int \sqrt{w}2(w-1)\, dw = 2\int w^{1/2}(w-1)\, dw \\ &= 2\int (w^{3/2} - w^{1/2})\, dw = 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + C \\ &= 2\left(\frac{2}{5}\left(1+\sqrt{x}\right)^{5/2} - \frac{2}{3}\left(1+\sqrt{x}\right)^{3/2}\right) + C .\end{aligned}$$

Notice that the substitution in the preceding example again converts the inside of the messiest function into something simple. In addition, since the derivative of the inside function is not waiting for us, we have to solve for *x* so that we can get dx entirely in terms of *w* and dw.

Example 14

Find $\int (x+7)\sqrt[3]{3-2x}\,dx$.

Solution

Here, instead of the derivative of the inside function (which is –2), we have the factor $(x+7)$. However, substituting $w = 3-2x$ turns out to help anyway. Then $dw = -2\,dx$, so $(-1/2)\,dw = dx$. Now we must convert everything to *w*, including $x+7$. If $w = 3-2x$, then $2x = 3-w$, so $x = 3/2 - w/2$, and therefore we can write $x+7$ in terms of *w*. Thus

$$\begin{aligned}
\int (x+7)\sqrt[3]{3-2x}\,dx &= \int \left(\frac{3}{2}-\frac{w}{2}+7\right)\sqrt[3]{w}\left(-\frac{1}{2}\right)dw \\
&= -\frac{1}{2}\int\left(\frac{17}{2}-\frac{w}{2}\right)w^{1/3}dw \\
&= -\frac{1}{4}\int(17-w)w^{1/3}dw \\
&= -\frac{1}{4}\int(17w^{1/3}-w^{4/3})\,dw \\
&= -\frac{1}{4}\left(17\frac{w^{4/3}}{4/3}-\frac{w^{7/3}}{7/3}\right)+C \\
&= -\frac{1}{4}\left(\frac{51}{4}(3-2x)^{4/3}-\frac{3}{7}(3-2x)^{7/3}\right)+C\,.
\end{aligned}$$

Looking back over the solution, the reason this substitution works is that it converts $\sqrt[3]{3-2x}$, the messiest part of the integrand, to $\sqrt[3]{w}$, which can be combined with the other term and then integrated.

Exercises and Problems for Section 7.1

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Exercises

1. Use substitution to express each of the following integrals as a multiple of $\int_a^b (1/w)\,dw$ for some *a* and *b*. Then evaluate the integrals.

- **(a).** $\int_0^1 \frac{x}{1+x^2}dx$
- **(b).** $\int_0^{\pi/4} \frac{\sin x}{\cos x}dx$

2.

- **(a).** Find the derivatives of $\sin(x^2+1)$ and $\sin(x^3+1)$.
- **(b).** Use your answer to part (a) to find antiderivatives of:
  - **(i).** $x\cos(x^2+1)$
  - **(ii).** $x^2\cos(x^3+1)$
- **(c).** Find the general antiderivatives of:
  - **(i).** $x\sin(x^2+1)$
  - **(ii).** $x^2\sin(x^3+1)$

Find the integrals in Exercises 3–44. Check your answers by differentiation.

3. $\int e^{3x}dx$

4. $\int e^{-x}dx$

5. $\int 25e^{-0.2t}dt$

6. $\int t \cos(t^2)\, dt$

7. $\int \sin(2x)\, dx$

8. $\int \sin(3-t)\, dt$

9. $\int x e^{-x^2} dx$

10. $\int y(y^2+5)^8 dy$

11. $\int t^2(t^3-3)^{10} dt$

12. $\int x^2(1+2x^3)^2 dx$

13. $\int x(x^2+3)^2 dx$

14. $\int x(x^2-4)^{7/2} dx$

15. $\int y^2(1+y)^2 dy$

16. $\int (2t-7)^{73} dt$

17. $\int \frac{dy}{y+5}$

18. $\int \frac{1}{\sqrt{4-x}} dx$

19. $\int (x^2+3)^2 dx$

20. $\int x^2 e^{x^3+1} dx$

21. $\int \sin\theta(\cos\theta+5)^7\, d\theta$

22. $\int \sqrt{\cos 3t}\sin 3t\, dt$

23. $\int \sin^6\theta\cos\theta\, d\theta$

24. $\int \sin^3\alpha\cos\alpha\, d\alpha$

25. $\int \sin^6(5\theta)\cos(5\theta)\, d\theta$

26. $\int \tan(2x)\, dx$

27. $\int \frac{(\ln z)^2}{z} dz$

28. $\int \frac{e^t+1}{e^t+t}\, dt$

29. $\int \frac{y}{y^2+4}\, dy$

30. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

31. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}} dy$

32. $\int \frac{1+e^x}{\sqrt{x+e^x}}dx$

33. $\int \frac{e^x}{2+e^x}dx$

34. $\int \frac{x+1}{x^2+2x+19}dx$

35. $\int \frac{t}{1+3t^2}dt$

36. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

37. $\int \frac{(t+1)^2}{t^2}dt$

38. $\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}dx$

39. $\int \cosh x\, dx$

40. $\int \sinh 3t\, dt$

41. $\int (\sinh z)e^{\cosh z}\, dz$

42. $\int \cosh(2w+1)\, dw$

43. $\int x \cosh x^2\, dx$

44. $\int \cosh^2 x \sinh x \, dx$

For the functions in Exercises 45–52, find the general antiderivative. Check your answers by differentiation.

45. $p(t) = \pi t^3 + 4t$

46. $f(x) = \sin 3x$

47. $f(x) = 2x \cos(x^2)$

48. $r(t) = 12t^2 \cos(t^3)$

49. $f(x) = \sin(2 - 5x)$

50. $f(x) = e^{\sin x} \cos x$

51. $f(x) = \frac{x}{x^2 + 1}$

52. $f(x) = \frac{1}{3\cos^2(2x)}$

For Exercises 53–60, use the Fundamental Theorem to calculate the definite integrals.

53. $\int_0^{\pi} \cos(x + \pi) \, dx$

54. $\int_0^{1/2} \cos(\pi x) \, dx$

55. $\int_0^{\pi/2} e^{-\cos \theta} \sin \theta \, d\theta$

56. $\int_1^2 2xe^{x^2} dx$

57. $\int_{1}^{8} \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$

58. $\int_{-1}^{e-2} \frac{1}{t+2} dt$

59. $\int_{1}^{4} \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$

60. $\int_{0}^{2} \frac{x}{(1+x^2)^2}\, dx$

For Exercises 61–66, evaluate the definite integrals. Whenever possible, use the Fundamental Theorem of Calculus, perhaps after a substitution. Otherwise, use numerical methods.

61. $\int_{-1}^{3} (x^3 + 5x)\, dx$

62. $\int_{-1}^{1} \frac{1}{1+y^2} dy$

63. $\int_{1}^{3} \frac{1}{x} dx$

64. $\int_{1}^{3} \frac{dt}{(t+7)^2}$

65. $\int_{-1}^{2} \sqrt{x+2}\, dx$

66. $\int_{1}^{2} \frac{\sin t}{t}\, dt$

Find the integrals in Exercises 67–74.

67. $\int y\sqrt{y+1}\,dy$

68. $\int z(z+1)^{1/3}dz$

69. $\int \frac{t^2+t}{\sqrt{t+1}}dt$

70. $\int \frac{dx}{2+2\sqrt{x}}$

71. $\int x^2\sqrt{x-2}\,dx$

72. $\int (z+2)\sqrt{1-z}\;dz$

73. $\int \frac{t}{\sqrt{t+1}}\,dt$

74. $\int \frac{3x-2}{\sqrt{2x+1}}\,dx$

Problems

75. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

(a). $\int x \sin(x^2)\, dx$

(b). $\int x^2 \sin x\, dx$

(c). $\int \frac{x^2}{1+x^2} dx$

(d). $\int \frac{x}{(1+x^2)^2} dx$

(e). $\int x^3 e^{x^2} dx$

(f). $\int \frac{\sin x}{2+\cos x}\, dx$

76. Suppose $\int_0^2 g(t)\, dt = 5$. Calculate the following:

(a). $\int_0^4 g(t/2)\, dt$

(b). $\int_0^2 g(2-t)\, dt$

77. Suppose $\int_0^1 f(t)\, dt = 3$. Calculate the following:

(a). $\int_0^{0.5} f(2t)\, dt$

(b). $\int_0^1 f(1-t)\, dt$

(c). $\int_0^{1.5} f(3-2t)\, dt$

78.

(a). Calculate exactly: $\int_{-\pi}^{\pi} \cos^2\theta \sin\theta \, d\theta$

(b). Calculate the exact area under the curve $y = \cos^2\theta \sin\theta$ between $\theta = 0$ and $\theta = \pi$.

79. Use the Fundamental Theorem to find the area under the graph of $f(x) = 1/(x+1)$ between $x = 0$ and $x = 2$.

80. Find the exact area under the graph of $f(x) = xe^{x^2}$ between $x = 0$ and $x = 2$.

81. Find the exact value of the area under one arch of the curve $V(t) = V_0 \sin(\omega t)$, where $V_0 > 0$ and $\omega > 0$.

82. Find the exact average value of $f(x) = 1/(x+1)$ on the interval $x = 0$ to $x = 2$. Sketch a graph showing the function and the average value.

83. Find $\int 4x(x^2+1)\, dx$ using two methods:

(a). Do the multiplication first, and then antidifferentiate.

(b). Use the substitution $w = x^2 + 1$.

(c). Explain how the expressions from parts (a) and (b) are different. Are they both correct?

84.

(a). Find $\int \sin\theta \cos\theta \, d\theta$.

(b). You probably solved part (a) by making the substitution $w = \sin\theta$ or $w = \cos\theta$. (If not, go back and do it that way.) Now find $\int \sin\theta \cos\theta \, d\theta$ by making the *other* substitution.

(c). There is yet another way of finding this integral which involves the trigonometric identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta .$$

Find $\int \sin\theta \cos\theta \, d\theta$ using one of these identities and then the substitution $w = 2\theta$.

(d). You should now have three different expressions for the indefinite integral $\int \sin\theta \cos\theta \, d\theta$. Are they really different? Are they all correct? Explain.

85. Let $I_{m,n} = \int_0^1 x^m (1-x)^n dx$ for constant *m*, *n*. Show that $I_{m,n} = I_{n,m}$.

86. If *t* is in years since 1990, the population, *P*, of the world in billions can be modeled by $P = 5.3e^{0.014t}$.

(a). What does this model give for the world population in 1990? In 2000?

(b). Use the Fundamental Theorem to find the average population of the world during the 1990s.

87. Oil is leaking out of a ruptured tanker at the rate of $r(t) = 50e^{-0.02t}$ thousand liters per minute.

(a). At what rate, in liters per minute, is oil leaking out at $t = 0$? At $t = 60$?

(b). How many liters leak out during the first hour?

88. Throughout much of the 20th century, the yearly consumption of electricity in the US increased exponentially at a continuous rate of 7% per year. Assume this trend continues and that the electrical energy consumed in 1900 was 1.4 million megawatt-hours.

(a). Write an expression for yearly electricity consumption as a function of time, *t*, in years since 1900.

(b). Find the average yearly electrical consumption throughout the 20th century.

(c). During what year was electrical consumption closest to the average for the century?

(d). Without doing the calculation for part (c), how could you have predicted which half of the century the answer would be in?

89. If we assume that wind resistance is proportional to velocity, then the downward velocity, *v*, of a body of mass *m* falling vertically is given by

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right),$$

where *g* is the acceleration due to gravity and *k* is a constant. Find the height, *h*, above the surface of the earth as a function of time. Assume the body starts at height h_0.

90. If we assume that wind resistance is proportional to the square of velocity, then the downward velocity, *v*, of a falling body is given by

$$v = \sqrt{\frac{g}{k}}\left(\frac{e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}\right).$$

Use the substitution $w = e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}$ to find the height, *h*, of the body above the surface of the earth as a function of time. Assume the body starts at a height h_0.

91.

(a). During the 1970s, ACME Widgets sold at a continuous rate of $R = R_0 e^{0.15t}$ widgets per year, where *t* is time in years since January 1, 1970. Suppose they were selling widgets at a rate of 1000 per year on the first day of the decade. How many widgets did they sell during the decade? How many did they sell if the rate on January 1, 1970 was 150,000,000 widgets per year?

(b). In the first case above (1000 widgets per year on January 1, 1970), how long did it take for half the widgets in the 1970s to be sold? In the second case (150,000,000 widgets per year on January 1, 1970), when had half the widgets in the 1970s been sold?

(c). In 1980 ACME began an advertising campaign claiming that half the widgets it had sold in the previous ten years were still in use. Based on your answer to part (b), about how long must a widget last in order to justify this claim?
