7.2 Integration by Parts

The method of substitution reverses the chain rule. Now we introduce *integration by parts*, which is based on the product rule.

Example 1

Find $\int xe^x dx$.

Solution

We are looking for a function whose derivative is xe^x. The product rule might lead us to guess xe^x, because we know that the derivative has two terms, one of which is xe^x:

$$\frac{d}{dx}(xe^x) = \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x) = e^x + xe^x .$$

Of course, our guess is wrong because of the extra e^x. But we can adjust our guess by subtracting e^x; this leads us to try $xe^x - e^x$. Let's check it:

$$\frac{d}{dx}(xe^x - e^x) = \frac{d}{dx}(xe^x) - \frac{d}{dx}(e^x) = e^x + xe^x - e^x = xe^x .$$

It works, so $\int xe^x \, dx = xe^x - e^x + C$.

Example 2

Find $\int \theta \cos\theta \, d\theta$.

Solution

We guess the antiderivative is $\theta \sin\theta$ and use the product rule to check:

$$\frac{d}{d\theta}(\theta \sin\theta) = \frac{d(\theta)}{d\theta} \sin\theta + \theta \frac{d}{d\theta}(\sin\theta) = \sin\theta + \theta\cos\theta .$$

To correct for the extra $\sin\theta$ term, we must subtract from our original guess something whose derivative is $\sin\theta$. Since $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$, we try:

$$\frac{d}{d\theta}(\theta \sin\theta + \cos\theta) = \frac{d}{d\theta}(\theta\sin\theta) + \frac{d}{d\theta}(\cos\theta) = \sin\theta + \theta\cos\theta - \sin\theta = \theta\cos\theta .$$

Thus, $\int \theta\cos\theta \, d\theta = \theta\sin\theta + \cos\theta + C$.

The General Formula for Integration by Parts

We can formalize the process illustrated in the last two examples in the following way. We begin with the product rule:

$$\frac{d}{dx}(uv) = u'v + uv'$$

where u and v are functions of x with derivatives u' and v', respectively. We rewrite this as:

$$uv' = \frac{d}{dx}(uv) - u'v$$

and then integrate both sides:

$$\int uv' dx = \int \frac{d}{dx}(uv) dx - \int u'v \, dx .$$

Since an antiderivative of $\frac{d}{dx}(uv)$ is just uv, we get the following formula:

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$$\int uv'dx = uv - \int u'v\,dx\,.$$

This formula is useful when the integrand can be viewed as a product and when the integral on the right-hand side is simpler than that on the left. In effect, we were using integration by parts in the previous two examples. In Example 1, we let $xe^x = (x)\cdot(e^x) = uv'$, and choose $u = x$ and $v' = e^x$. Thus, $u' = 1$ and $v = e^x$, so

$$\int \underbrace{(x)}_{u}\underbrace{(e^x)}_{v'}dx = \underbrace{(x)}_{u}\underbrace{(e^x)}_{v} - \int \underbrace{(1)}_{u'}\underbrace{(e^x)}_{v}dx = xe^x - e^x + C\,.$$

So uv represents our first guess, and $\int u'v\,dx$ represents the correction to our guess.

Notice what would have happened if we took $v = e^x + C_1$. Then

$$\begin{aligned}\int xe^x dx &= x(e^x + C_1) - \int (e^x + C_1)dx \\ &= xe^x + C_1x - e^x - C_1x + C \\ &= xe^x - e^x + C,\end{aligned}$$

as before. Thus, it is not necessary to include an arbitrary constant in the antiderivative for *v*; any antiderivative will do.

What would have happened if we had picked *u* and v' the other way around? If $u = e^x$ and $v' = x$, then $u' = e^x$ and $v = x^2/2$. The formula for integration by parts then gives

$$\int xe^x dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}\cdot e^x dx,$$

which is true but not helpful since the integral on the right is worse than the one on the left. To use this method, we must choose *u* and v' to make the integral on the right easier to find than the integral on the left.

How to Choose u and v'

- Whatever you let v' be, you need to be able to find *v*.
- It helps if u' is simpler than *u* (or at least no more complicated than *u*).
- It helps if *v* is simpler than v' (or at least no more complicated than v').

If we pick $v' = x$ in Example 1, then $v = x^2 / 2$, which is certainly “worse” than v'.

There are some examples which don't look like good candidates for integration by parts because they don't appear to involve products, but for which the method works well. Such examples often involve $\ln x$ or the inverse trigonometric functions. Here is one:

Example 3

Find $\int_2^3 \ln x \, dx$.

Solution

This does not look like a product unless we write $\ln x = (1)(\ln x)$. Then we might say $u = 1$ so $u' = 0$, which certainly makes things simpler. But if $v' = \ln x$, what is *v*? If we knew, we would not need integration by parts. Let's try the other way: if $u = \ln x$, $u' = 1 / x$ and if $v' = 1$, $v = x$, so

$$\int_2^3 \underbrace{(\ln x)}_{u}\underbrace{(1)}_{v'}\,dx = \underbrace{(\ln x)}_{u}\underbrace{(x)}_{v}\Big|_2^3 - \int_2^3 \underbrace{\left(\frac{1}{x}\right)}_{u'}\cdot\underbrace{(x)}_{v}\,dx$$

$$= x\ln x\Big|_2^3 - \int_2^3 1\,dx = (x\ln x - x)\big|_2^3$$

$$= 3\ln 3 - 3 - 2\ln 2 + 2 = 3\ln 3 - 2\ln 2 - 1\,.$$

Notice that when doing a definite integral by parts, we must remember to put the limits of integration (here 2 and 3) on the uv term (in this case $x\ln x$) as well as on the integral $\int u'v\,dx$.

Example 4

Find $\int x^6 \ln x\,dx$.

Solution

View $x^6 \ln x$ as uv' where $u = \ln x$ and $v' = x^6$. Then $v = \frac{1}{7}x^7$ and $u' = 1/x$, so integration by parts gives us:

$$\int x^6 \ln x\,dx = \int (\ln x)x^6 dx = (\ln x)\left(\frac{1}{7}x^7\right) - \int \frac{1}{7}x^7\cdot\frac{1}{x}dx$$

$$= \frac{1}{7}x^7\ln x - \frac{1}{7}\int x^6 dx$$

$$= \frac{1}{7}x^7\ln x - \frac{1}{49}x^7 + C\,.$$

In Example 4 we did not choose $v' = \ln x$, because it is not immediately clear what v would be. In fact, we used integration by parts in Example 3 to find the antiderivative of $\ln x$. Also, using $u = \ln x$, as we have done, gives $u' = 1/x$, which can be considered simpler than $u = \ln x$. This shows that u does not have to be the first factor in the integrand (here x^6).

Example 5

Find $\int x^2 \sin 4x\, dx$.

Solution

If we let $v' = \sin 4x$, then $v = -\frac{1}{4}\cos 4x$, which is no worse than v'. Also letting $u = x^2$, we get $u' = 2x$, which is simpler than $u = x^2$. Using integration by parts:

$$\begin{aligned}\int x^2 \sin 4x\, dx &= x^2\left(-\frac{1}{4}\cos 4x\right) - \int 2x\left(-\frac{1}{4}\cos 4x\right)dx\\ &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x\, dx\ .\end{aligned}$$

The trouble is we still have to grapple with $\int x\cos 4x\, dx$. This can be done by using integration by parts again with a new *u* and *v*, namely $u = x$ and $v' = \cos 4x$:

$$\begin{aligned}\int x\cos 4x\, dx &= x\left(\frac{1}{4}\sin 4x\right) - \int 1\cdot\frac{1}{4}\sin 4x\, dx\\ &= \frac{1}{4}x\sin 4x - \frac{1}{4}\cdot\left(-\frac{1}{4}\cos 4x\right) + C\\ &= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C\ .\end{aligned}$$

Thus,

$$\begin{aligned}\int x^2\sin 4x\, dx &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x\, dx\\ &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\left(\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C\right)\\ &= -\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C\ .\end{aligned}$$

Notice that, in this example, each time we used integration by parts, the exponent of *x* went down by 1. In addition, when the arbitrary constant *C* is multiplied by $\frac{1}{2}$, it is still represented by *C*.

Example 6

Find $\int \cos^2\theta \, d\theta$.

Solution

Using integration by parts with $u = \cos\theta$, $v' = \cos\theta$ gives $u' = -\sin\theta$, $v = \sin\theta$, so we get

$$\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int \sin^2\theta \, d\theta$$

Substituting $\sin^2\theta = 1 - \cos^2\theta$ leads to

$$\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int (1 - \cos^2\theta) d\theta$$

$$= \cos\theta \sin\theta + \int 1 \, d\theta - \int \cos^2\theta \, d\theta .$$

Looking at the right side, we see that the original integral has reappeared. If we move it to the left, we get

$$2\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int 1 \, d\theta = \cos\theta \sin\theta + \theta + C .$$

Dividing by 2 gives

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\cos\theta \sin\theta + \frac{1}{2}\theta + C .$$

Problem 41 asks you to do this integral by another method.

The previous example illustrates a useful technique : Use integration by parts to transform the integral into an expression containing another copy of the same integral, possibly multiplied by a coefficient, then solve for the original integral. It may be necessary to apply integration by parts twice, as the next example shows.

Example 7

Use integration by parts twice to find $\int e^{2x} \sin(3x)\, dx$.

Solution

Using integration by parts with $u = e^{2x}$ and $v' = \sin(3x)$ gives $u' = 2e^{2x}, v = -\frac{1}{3}\cos(3x)$, so we get

$$\int e^{2x} \sin(3x)\, dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)\, dx .$$

On the right side we have an integral similar to the original one, with the sine replaced by a cosine. Using integration by parts on that integral in the same way gives

$$\int e^{2x}\cos(3x)\, dx = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\, dx.$$

Substituting this into the expression we obtained for the original integral gives

$$\begin{aligned}\int e^{2x}\sin(3x)\, dx &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\, dx\right)\\ &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\int e^{2x}\sin(3x)dx\end{aligned}$$

The right side now has a copy of the original integral, multiplied by $-4/9$. Moving it to the left, we get

$$\left(1+\frac{4}{9}\right)\int e^{2x}\sin(3x)\, dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) .$$

Dividing through by the coefficient on the left, $(1 + 4/9) = 13/9$, we get

$$\begin{aligned}\int e^{2x}\sin(3x)\, dx &= \frac{9}{13}\left(-\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)\right)\\ &= \frac{1}{13}e^{2x}(2\sin(3x) - 3\cos(3x)) + C .\end{aligned}$$

Exercises and Problems for Section 7.2

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Exercises

1. Write $\arctan x = 1 \cdot \arctan x$ to find $\int \arctan x \, dx$.

Find the integrals in Exercises 2–29.

2. $\int t \sin t \, dt$

3. $\int t^2 \sin t \, dt$

4. $\int te^{5t} \, dt$

5. $\int t^2 e^{5t} \, dt$

6. $\int pe^{-0.1p} \, dp$

7. $\int (z+1)e^{2z} \, dz$

8. $\int y \ln y \, dy$

9. $\int x^3 \ln x \, dx$

10. $\int q^5 \ln 5q \, dq$

11. $\int \theta^2 \cos 3\theta \, d\theta$

12. $\int \sin^2\theta \, d\theta$

13. $\int \cos^2(3\alpha + 1) \, d\alpha$

14. $\int (\ln t)^2 \, dt$

15. $\int y\sqrt{y+3} \, dy$

16. $\int (t+2)\sqrt{2+3t} \, dt$

17. $\int (\theta+1)\sin(\theta+1) \, d\theta$

18. $\int \frac{z}{e^z} dz$

19. $\int \frac{\ln x}{x^2} dx$

20. $\int \frac{y}{\sqrt{5-y}} dy$

21. $\int \frac{t+7}{\sqrt{5-t}} dt$

22. $\int x(\ln x)^4\,dx$

23. $\int \arcsin w\,dw$

24. $\int \arctan 7z\,dz$

25. $\int x \arctan x^2\,dx$

26. $\int x^3 e^{x^2}\,dx$

27. $\int x^5 \cos x^3\,dx$

28. $\int x \sinh x\,dx$

29. $\int (x-1)\cosh x\,dx$

Evaluate the integrals in Exercises 30–37 both exactly [e.g. $\ln(3\pi)$] and numerically [e.g. $\ln(3\pi) \approx 2.243$]:

30. $\int_1^5 \ln t\,dt$

31. $\int_3^5 x \cos x\,dx$

32. $\int_0^{10} z e^{-z}\,dz$

33. $\int_1^3 t \ln t \, dt$

34. $\int_0^1 \arctan y \, dy$

35. $\int_0^5 \ln(1+t) \, dt$

36. $\int_0^1 \arcsin z \, dz$

37. $\int_0^1 u \arcsin u^2 \, du$

Problems

38. For each of the following integrals, indicate whether integration by substitution or integration by parts is more appropriate. Do not evaluate the integrals.

(a). $\int x \sin x \, dx$

(b). $\int \frac{x^2}{1+x^3} \, dx$

(c). $\int x e^{x^2} \, dx$

(d). $\int x^2 \cos(x^3) \, dx$

(e). $\int \frac{1}{\sqrt{3x+1}} \, dx$

(f). $\int x^2 \sin x \, dx$

(g). $\int \ln x \, dx$

39. Find the exact value of the area under the first arch of $f(x) = x \sin x$.

40. In Exercise 12, you evaluated $\int \sin^2\theta \, d\theta$ using integration by parts. (If you did not do it by parts, do so now!) Redo this integral using the identity $\sin^2\theta = (1 - \cos 2\theta) / 2$. Explain any differences in the form of the answer obtained by the two methods.

41. Compute $\int \cos^2\theta \, d\theta$ in two different ways and explain any differences in the form of your answers. (The identity $\cos^2\theta = (1 + \cos 2\theta) / 2$ may be useful.)

42. Use integration by parts twice to find $\int e^x \sin x \, dx$.

43. Use integration by parts twice to find $\int e^\theta \cos \theta \, d\theta$.

44. Use the results from Problems 42 and 43 and integration by parts to find $\int xe^x \sin x \, dx$.

45. Use the results from Problems 42 and 43 and integration by parts to find $\int \theta e^\theta \cos \theta \, d\theta$.

In Problems 46–49, derive the given formulas.

46. $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x dx$

47. $\int x^n \cos ax \, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx$

48. $\int x^n \sin ax \, dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$

49. $\displaystyle\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

50. Integrating $e^{ax}\sin bx$ by parts twice yields a result of the form

$$\int e^{ax}\sin bx\, dx = e^{ax}(A\sin bx + B\cos bx) + C .$$

(a). Find the constants *A* and *B* in terms of *a* and *b*. [Hint: Don't actually perform the integration by parts.]

(b). Evaluate $\int e^{ax}\cos bx\, dx$ by modifying the result in part (a). [Again, it is not necessary to perform integration by parts, as the result is of the same form as that in part (a).]

51. Estimate $\displaystyle\int_0^{10} f(x)g'(x)\, dx$ if $f(x) = x^2$ and *g* has the values in the following table.

x	0	2	4	6	8	10
$g(x)$	2.3	3.1	4.1	5.5	5.9	6.1

52. Let *f* be twice differentiable with $f(0) = 6$, $f(1) = 5$, and $f'(1) = 2$. Evaluate the integral $\displaystyle\int_0^1 x f''(x)\, dx$.

53. Let $F(a)$ be the area under the graph of $y = x^2 e^{-x}$ between $x = 0$ and $x = a$, for $a > 0$.

(a). Find a formula for $F(a)$.

(b). Is *F* an increasing or decreasing function?

(c). Is *F* concave up or concave down for $0 < a < 2$?

54. The concentration, *C*, in ng/ml, of a drug in the blood as a function of the time, *t*, in hours since the drug was administered is given by $C = 15te^{-0.2t}$. The area under the concentration curve is a measure of the overall effect of the drug on the body, called the bioavailability. Find the bioavailability of the drug between $t = 0$ and $t = 3$.

55. The voltage, *V*, in an electric circuit is given as a function of time, *t*, by

$$V = V_0 \cos(\omega t + \phi) .$$

Suppose each of the positive constants, V_0, ω, ϕ is increased (while the other two are held constant). What is the effect of each increase on the following quantities:

(a). The maximum value of *V*?

(b). The maximum value of dV / dt?

(c). The average value of V^2 over one period?

56. During a surge in the demand for electricity, the rate, *r*, at which energy is used can be approximated by

$$r = te^{-at},$$

where *t* is the time in hours and *a* is a positive constant.

(a). Find the total energy, *E*, used in the first *T* hours. Give your answer as a function of *a*.

(b). What happens to *E* as $T \to \infty$?

57. In describing the behavior of an electron, we use wave functions $\Psi_1, \Psi_2, \Psi_3, \cdots$ of the form

$$\Psi_n(x) = C_n \sin(n\pi x) \qquad for\ n = 1, 2, 3, \ldots$$

where *x* is the distance from a fixed point and C_n is a positive constant.

(a). Find C_1 so that Ψ_1 satisfies

$$\int_0^1 (\Psi_1(x))^2\, dx = 1 .$$

This is called normalizing the wave function Ψ_1.

(b). For any integer *n*, find C_n so that Ψ_n is normalized.