7.3 Tables of Integrals

Since so few functions have elementary antiderivatives, they have been compiled in a list called a table of integrals.[1] A short table of indefinite integrals. The key to using these tables is being able to recognize the general class of function that you are trying to integrate, so you can know in what section of the table to look.

Warning: This section involves long division of polynomials and completing the square. You may want to review these topics!

Using the Table of Integrals

Part I of the table of indefinite integrals gives the antiderivatives of the basic functions x^n, a^x, $\ln x$, $\sin x$, $\cos x$, and $\tan x$. (The antiderivative for $\ln x$ is found using integration by parts and is a special case of the more general formula III-13.) Most of these are already familiar.

Part II of the table contains antiderivatives of functions involving products of e^x, $\sin x$, and $\cos x$. All of these antiderivatives were obtained using integration by parts.

Example 1

Find $\int \sin 7z \sin 3z \, dz$.

Solution

Since the integrand is the product of two sines, we should use II-10 in the table,

$$\int \sin 7z \sin 3z \, dz = -\frac{1}{40}(7\cos 7z \sin 3z - 3\cos 3z \sin 7z) + C.$$

Part III of the table contains antiderivatives for products of a polynomial and e^x, $\sin x$, or $\cos x$. It also has an antiderivative for $x^n \ln x$, which can easily be used to find the antiderivatives of the product of a general polynomial and $\ln x$. Each *reduction formula* is used repeatedly to reduce the degree of the polynomial until a zero degree polynomial is obtained.

Example 2

Find $\int (x^5 + 2x^3 - 8)e^{3x} \, dx$.

Solution

Since $p(x) = x^5 + 2x^3 - 8$ is a polynomial multiplied by e^{3x}, this is of the form in III-14. Now $p'(x) = 5x^4 + 6x^2$ and $p''(x) = 20x^3 + 12x$, and so on, giving

$$\int (x^5 + 2x^3 - 8)e^{3x} \, dx = e^{3x}\left[\frac{1}{3}(x^5 + 2x^3 - 8) - \frac{1}{9}(5x^4 + 6x^2) + \frac{1}{27}(20x^3 + 12x) - \frac{1}{81}(60x^2 + 12) + \frac{1}{243}(120x) - \frac{1}{729} \cdot 120\right] + C.$$

Here we have the successive derivatives of the original polynomial $x^5 + 2x^3 - 8$, occurring with alternating signs and multiplied by successive powers of 1/3.

Part IV of the table contains reduction formulas for the antiderivatives of $\cos^n x$ and $\sin^n x$, which can be obtained by integration by parts. When *n* is a positive integer, formulas IV-17 and IV-18 can be used repeatedly to reduce the power *n* until it is 0 or 1.

Example 3

Find $\int \sin^6\theta \, d\theta$.

Solution

Use IV-17 repeatedly:

$$\int \sin^6\theta \, d\theta = -\frac{1}{6} \sin^5 \theta \cos \theta + \frac{5}{6} \int \sin^4\theta \, d\theta$$

$$\int \sin^4\theta \, d\theta = -\frac{1}{4} \sin^3 \theta \cos \theta + \frac{3}{4} \int \sin^2\theta \, d\theta$$

$$\int \sin^2\theta \, d\theta = -\frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \int 1 \, d\theta \, .$$

Calculate $\int \sin^2\theta \, d\theta$ first, and use this to find $\int \sin^4\theta \, d\theta$; then calculate $\int \sin^6\theta \, d\theta$. Putting this all together, we get

$$\int \sin^6\theta \, d\theta = -\frac{1}{6}\sin^5\theta \cos \theta - \frac{5}{24}\sin^3\theta \cos \theta - \frac{15}{48}\sin \theta \cos \theta + \frac{15}{48}\theta + C \, .$$

The last item in **Part IV** of the table is not a formula: it is advice on how to antidifferentiate products of integer powers of $\sin x$ and $\cos x$. There are various techniques to choose from, depending on the nature (odd or even, positive or negative) of the exponents.

Example 4

Find $\int \cos^3 t \, \sin^4 t \, dt$.

Solution

Here the exponent of $\cos t$ is odd, so IV-23 recommends making the substitution $w = \sin t$. Then $dw = \cos t \, dt$. To make this work, we'll have to separate off one of the cosines to be part of dw. Also, the remaining even power of $\cos t$ can be rewritten in terms of $\sin t$ by using $\cos^2 t = 1 - \sin^2 t = 1 - w^2$, so that

$$\int \cos^3 t \sin^4 t \, dt = \int \cos^2 t \sin^4 t \cos t \, dt$$
$$= \int (1 - w^2) w^4 \, dw = \int (w^4 - w^6) \, dw$$
$$= \frac{1}{5} w^5 - \frac{1}{7} w^7 + C = \frac{1}{5} \sin^5 t - \frac{1}{7} \sin^7 t + C .$$

Example 5

Find $\int \cos^2 x \sin^4 x \, dx$.

Solution

In this example, both exponents are even. The advice given in IV-23 is to convert to all sines or all cosines. We'll convert to all sines by substituting $\cos^2 x = 1 - \sin^2 x$, and then we'll multiply out the integrand:

$$\int \cos^2 x \sin^4 x \, dx = \int (1 - \sin^2 x) \sin^4 x \, dx = \int \sin^4 x \, dx - \int \sin^6 x \, dx .$$

In Example 3 we found $\int \sin^4 x \, dx$ and $\int \sin^6 x \, dx$. Put them together to get

$$\int \cos^2 x \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x$$
$$- \left(-\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{15}{48} \sin x \cos x + \frac{15}{48} x \right) + C$$
$$= \frac{1}{6} \sin^5 x \cos x - \frac{1}{24} \sin^3 x \cos x - \frac{3}{48} \sin x \cos x + \frac{3}{48} x + C .$$

The last two parts of the table are concerned with quadratic functions: **Part V** has expressions with quadratic denominators; **Part VI** contains square roots of quadratics. The quadratics that appear in these formulas are of the form $x^2 \pm a^2$ or $a^2 - x^2$, or in factored form $(x - a)(x - b)$, where *a* and *b* are different constants. Quadratics can be converted to these forms by factoring or completing the square.

Preparing to Use the Table: Transforming the Integrand

To use the integral table, we often need to manipulate or reshape integrands to fit entries in the table. The manipulations that tend to be useful are factoring, long division, completing the square, and substitution.

Using Factoring

Example 6

Find $\int \frac{3x+7}{x^2+6x+8}dx$.

Solution

In this case we factor the denominator to get it into a form in the table:

$$x^2+6x+8=(x+2)(x+4).$$

.

Now in V-27 we let $a=-2$, $b=-4$, $c=3$, and $d=7$, to obtain

$$\int \frac{3x+7}{x^2+6x+8}dx=\frac{1}{2}(\ln|x+2|-(-5)\ln|x+4|)+C.$$

Long Division

Example 7

Find $\int \frac{x^2}{x^2+4}dx$.

Solution

A good rule of thumb when integrating a rational function whose numerator has a degree greater than or equal to that of the denominator is to start by doing *long division*. This results in a polynomial plus a simpler rational function as a remainder. Performing long division here, we obtain:

$$\frac{x^2}{x^2+4}=1-\frac{4}{x^2+4}.$$

Then, by V-24 with $a = 2$, we obtain:

$$\int \frac{x^2}{x^2+4} dx = \int 1\, dx - 4 \int \frac{1}{x^2+4} dx = x - 4 \cdot \frac{1}{2} \arctan \frac{x}{2} + C = x - 2 \arctan \frac{x}{2} + C .$$

Completing the Square to Rewrite the Quadratic in the Form $w^2 + a^2$

Example 8

Find $\int \frac{1}{x^2 + 6x + 14} dx$.

Solution

By completing the square, we can get this integrand into a form in the table:

$$\begin{aligned} x^2 + 6x + 14 &= (x^2 + 6x + 9) - 9 + 14 \\ &= (x+3)^2 + 5 . \end{aligned}$$

Let $w = x + 3$. Then $dw = dx$ and so the substitution gives

$$\int \frac{1}{x^2+6x+14} dx = \int \frac{1}{w^2+5} dw = \frac{1}{\sqrt{5}} \arctan \frac{w}{\sqrt{5}} + C = \frac{1}{\sqrt{5}} \arctan \frac{x+3}{\sqrt{5}} + C,$$

where the antidifferentiation uses V-24 with $a^2 = 5$.

Substitution

Example 9

Find $\int e^t \sin(5t+7)\, dt$.

Solution

This looks similar to II-8. To make the correspondence more complete, let's try the substitution $w = 5t + 7$. Then $dw = 5\, dt$, so $dt = \frac{1}{5} dw$. Also, $t = (w-7)/5$. Then the integral becomes

$$\int e^t \sin(5t+7) dt = \int e^{(w-7)/5} \sin w \frac{dw}{5}$$

$$= \frac{e^{-7/5}}{5} \int e^{w/5} \sin w dw \qquad (\text{since } e^{(w-7)/5} = e^{w/5} e^{-7/5} \text{ and } e^{-7/5} \text{ is a constant})$$

Now we can use II-8 with $a = \frac{1}{5}$ and $b = 1$ to write

$$\int e^{w/5} \sin w\, dw = \frac{1}{\left(\frac{1}{5}\right)^2 + 1^2} e^{w/5} \left(\frac{\sin w}{5} - \cos w \right) + C,$$

so

$$\int e^t \sin(5t+7) dt = \frac{e^{-7/5}}{5} \left(\frac{25}{26} e^{(5t+7)/5} \left(\frac{\sin(5t+7)}{5} - \cos(5t+7) \right) \right) + C$$

$$= \frac{5e^t}{26} \left(\frac{\sin(5t+7)}{5} - \cos(5t+7) \right) + C\,.$$

Exercises and Problems for Section 7.3

Click here to open Student Solutions Manual: Ch 07 Section 03

Click here to open Web Quiz Ch 07 Section 03

Exercises

For Exercises 1–40, antidifferentiate using the table of integrals. You may need to transform the integrand first.

1. $\int e^{-3\theta}\cos\theta\, d\theta$

2. $\int x^5 \ln x\, dx$

3. $\int x^3 \sin 5x\, dx.$

4. $\int (x^2+3)\ln x\, dx.$

5. $\int (x^3+5)^2 dx.$

6. $\int \sin w \cos^4 w\, dw$

7. $\int \sin^4 x\, dx$

8. $\int \frac{1}{3+y^2} dy$

9. $\int x^3 e^{2x} dx$

10. $\int \frac{dx}{9x^2+16}$

11. $\int \frac{dx}{\sqrt{25-16x^2}}$

12. $\int \frac{dx}{\sqrt{9x^2+25}}$

13. $\int \sin 3\theta \cos 5\theta\, d\theta$

14. $\int \sin 3\theta \sin 5\theta \, d\theta$

15. $\int x^2 e^{3x} dx$

16. $\int x^2 e^{x^3} dx$

17. $\int x^4 e^{3x} dx$

18. $\int u^5 \ln(5u) \, du$

19. $\int \frac{1}{\cos^3 x} dx$

20. $\int \frac{t^2 + 1}{t^2 - 1} dt$

21. $\int x^3 \sin x^2 dx$

22. $\int \cos 2y \cos 7y \, dy$

23. $\int y^2 \sin 2y \, dy$

24. $\int e^{5x} \sin 3x \, dx$

25. $\int \frac{1}{\cos^5 x} dx.$

26. $\int \frac{1}{\sin^2 2\theta} d\theta$

27. $\int \frac{1}{\sin^3 3\theta} d\theta$

28. $\int \frac{1}{\cos^4 7x} dx$

29. $\int \frac{1}{x^2 + 4x + 3} dx$

30. $\int \tan^4 x \, dx$

31. $\int \frac{dz}{z(z-3)}$

32. $\int \frac{dy}{4 - y^2}$

33. $\int \frac{1}{1 + (z+2)^2} dz$

34. $\int \frac{1}{y^2 + 4y + 5} dy$

35. $\int \frac{1}{x^2 + 4x + 4} dx$

36. $\int \sin^3 x \, dx$

37. $\int \sin^3 3\theta \cos^2 3\theta \, d\theta$

38. $\int z e^{2z^2} \cos(2z^2) \, dz$

39. $\int \sinh^3 x \cosh^2 x \, dx$

40. $\int \sinh^2 x \cosh^3 x \, dx$

For Problems 41–44, evaluate the definite integrals. Whenever possible, use the Fundamental Theorem of Calculus, perhaps after a substitution. Otherwise, use numerical methods.

41. $\int_0^{\pi/12} \sin(3\alpha) d\alpha$

42. $\int_0^1 \frac{1}{x^2 + 2x + 1} dx$

43. $\int_0^1 \frac{(x+2)}{(x+2)^2 + 1} dx$

44. $\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1 - x^4}} dx$

Problems

45. Show that for all integers *m* and *n*, with $m \neq \pm n$, $\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = 0$.

46. Show that for all integers *m* and *n*, with $m \neq \pm n$, $\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = 0$.

47. The voltage, *V*, in an electrical outlet is given as a function of time, *t*, by the function $V = V_0 \cos(120\pi t)$, where *V* is in volts and *t* is in seconds, and V_0 is a positive constant representing the maximum voltage.

(a). What is the average value of the voltage over 1 second?

(b). Engineers do not use the average voltage. They use the root mean square voltage defined by $\bar{V} = \sqrt{\text{average of } (V^2)}$. Find $\bar{V}$ in terms of V_0. (Take the average over 1 second.)

(c). The standard voltage in an American house is 110 volts, meaning that $\bar{V} = 110$. What is V_0?

48. For some constants *A* and *B*, the rate of production, $R(t)$, of oil in a new oil well is modeled by:

$$R(t) = A + Be^{-t} \sin(2\pi t)$$

where t is the time in years, A is the equilibrium rate, and B is the "variability" coefficient.

(a). Find the total amount of oil produced in the first N years of operation. (Take N to be an integer.)

(b). Find the average amount of oil produced per year over the first N years (where N is an integer).

(c). From your answer to part (b), find the average amount of oil produced per year as $N \to \infty$.

(d). Looking at the function $R(t)$, explain how you might have predicted your answer to part (c) without doing any calculations.

(e). Do you think it is reasonable to expect this model to hold over a very long period? Why or why not?

49. For a positive integer n, let $\Psi_n(x) = C_n \sin(n\pi x)$ be the wave function used in describing the behavior of an electron. If n and m are different positive integers, find

$$\int_0^1 \Psi_n(x) \cdot \Psi_m(x)\, dx\,.$$