7.4 Algebraic Identities and Trigonometric Substitutions

Although not all functions have elementary antiderivatives, many do. In this section we introduce two powerful methods of integration which show that large classes of functions have elementary antiderivatives. The first is the method of partial fractions, which depends on an algebraic identity, and allows us to integrate rational functions. The second is the method of trigonometric substitutions, which allows us to handle expressions involving the square root of a quadratic polynomial. Some of the formulas in the table of integrals can be derived using the techniques of this section.

Method of Partial Fractions

The integral of some rational functions can be obtained by splitting the integrand into *partial fractions*. For example, to find

$$\int \frac{1}{(x-2)(x-5)}\,dx,$$

the integrand is split into partial fractions with denominators $(x-2)$ and $(x-5)$. We write

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5},$$

where A and B are constants that need to be found. Multiplying by $(x-2)(x-5)$ gives the identity

$$1 = A(x-5) + B(x-2)$$

so

$$1 = (A+B)x - 5A - 2B.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficients of x on both sides must be equal. So

$$-5A - 2B = 1$$
$$A + B = 0\,.$$

Solving these equations gives $A = -1/3, B = 1/3$. Thus,

$$\frac{1}{(x-2)(x-5)} = \frac{-1/3}{x-2} + \frac{1/3}{x-5}\,.$$

Example 1

Use partial fractions to integrate $\int \frac{1}{(x-2)(x-5)} dx$.

Solution

We split the integrand into partial fractions, each of which can be integrated:

$$\int \frac{1}{(x-2)(x-5)} dx = \int \left(\frac{-1/3}{x-2} + \frac{1/3}{x-5} \right) dx = -\frac{1}{3} \ln|x-2| + \frac{1}{3} \ln|x-5| + C\,.$$

You can check that using formula V-26 in the integral table gives the same result.

This method can be used to derive formulas V-26 and V-27 in the integral table. A similar method works whenever the denominator of the integrand factors into distinct linear factors and the numerator has degree less than the denominator.

Example 2

Find $\int \frac{x+2}{x^2+x} dx$.

Solution

We factor the denominator and split the integrand into partial fractions:

$$\frac{x+2}{x^2+x} = \frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}\,.$$

Multiplying by $x(x+1)$ gives the identity

$$x + 2 = A(x+1) + Bx$$
$$= (A+B)x + A\,.$$

Equating constant terms and coefficients of x gives $A = 2$ and $A + B = 1$, so $B = -1$. Then we split the integrand into two parts and integrate:

$$\int \frac{x+2}{x^2+x}dx = \int \left(\frac{2}{x} - \frac{1}{x+1}\right)dx = 2\ln|x| - \ln|x+1| + C.$$

The next example illustrates what to do if there is a repeated factor in the denominator.

Example 3

Calculate $\int \frac{10x-2x^2}{(x-1)^2(x+3)}dx$ using partial fractions of the form $\frac{A}{x-1}$, $\frac{B}{(x-1)^2}$, $\frac{C}{x+3}$.

Solution

We are given that the squared factor, $(x-1)^2$, leads to partial fractions of the form:

$$\frac{10x-2x^2}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

Multiplying through by $(x-1)^2(x+3)$ gives

$$\begin{aligned} 10x - 2x^2 &= A(x-1)(x+3) + B(x+3) + C(x-1)^2 \\ &= (A+C)x^2 + (2A+B-2C)x - 3A + 3B + C. \end{aligned}$$

Equating the coefficients of x^2 and x and the constant terms, we get the simultaneous equations:

$$\begin{aligned} A + C &= -2 \\ 2A + B - 2C &= 10 \\ -3A + 3B + C &= 0 \end{aligned}$$

Solving gives $A = 1, B = 2, C = -3$. Thus, we obtain three integrals which can be evaluated:

$$\begin{aligned} \int \frac{10x-2x^2}{(x-1)^2(x+3)}dx &= \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{3}{x+3}\right)dx \\ &= \ln|x-1| - \frac{2}{(x-1)} - 3\ln|x+3| + K. \end{aligned}$$

For the second integral, we use the fact that $\int 2/(x-1)^2 dx = 2\int (x-1)^{-2}\,dx = -2(x-1)^{-1} + K$.

If there is a quadratic in the denominator which cannot be factored, we must allow a numerator of the form $Ax + B$ in the numerator, as the next example shows.

Example 4

Find $\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)} dx$ using partial fractions of the form $\frac{Ax+B}{x^2+1}$ and $\frac{C}{x-2}$.

Solution

We are given that the quadratic denominator, (x^2+1), which cannot be factored further, has a numerator of the form $Ax+B$, so we have

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}.$$

Multiplying by $(x^2+1)(x-2)$ gives

$$\begin{aligned} 2x^2 - x - 1 &= (Ax+B)(x-2) + C(x^2+1) \\ &= (A+C)x^2 + (B-2A)x + C - 2B \end{aligned}$$

Equating the coefficients of x^2 and x and the constant terms gives the simultaneous equations

$$\begin{aligned} A + C &= 2 \\ B - 2A &= -1 \\ C - 2B &= -1. \end{aligned}$$

Solving gives $A = B = C = 1$, so we rewrite the integral as follows:

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)} dx = \int \left(\frac{x+1}{x^2+1} + \frac{1}{x-2} \right) dx.$$

This identity is useful provided we can perform the integration on the right-hand side. The first integral can be done if it is split into two; the second integral is similar to those in the previous examples. We have

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)} dx = \int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx + \int \frac{1}{x-2} dx.$$

To calculate $\int (x/(x^2+1)) dx$, substitute $w = x^2 + 1$, or guess and check. The final result is

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)} dx = \frac{1}{2}\ln\left|x^2+1\right| + \arctan x + \ln|x-2| + K.$$

Finally, the next example shows what to do if the numerator has degree larger than the denominator.

Example 5

Calculate $\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} dx$ using long division.

Solution

The degree of the numerator is greater than the degree of the denominator, so we divide first:

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = \frac{x(x^2 - 7x + 10) + 1}{x^2 - 7x + 10} = x + \frac{1}{x^2 - 7x + 10}.$$

The remainder, in this case $1 / (x^2 - 7x + 10)$, is a rational function on which we try to use partial fractions. We have

$$\frac{1}{x^2 - 7x + 10} = \frac{1}{(x - 2)(x - 5)}$$

so in this case we use the result of Example 1 to obtain

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} dx = \int \left(x + \frac{1}{(x - 2)(x - 5)} \right) dx = \frac{x^2}{2} - \frac{1}{3} \ln|x - 2| + \frac{1}{3} \ln|x - 5| + C.$$

Many, though not all, rational functions can be integrated by the strategy suggested by the previous examples.

Strategy for Integrating a Rational Function, $\frac{P(x)}{Q(x)}$

- If degree of $P(x) \geq$ degree of $Q(x)$, try long division and the method of partial fractions on the remainder.
- If $Q(x)$ is the product of distinct linear factors, use partial fractions of the form

$$\frac{A}{(x - c)}.$$

- If $Q(x)$ contains a repeated linear factor, $(x-c)^n$, use partial fractions of the form

$$\frac{A_1}{(x-c)} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n}.$$

- If $Q(x)$ contains an unfactorable quadratic $q(x)$, try a partial fraction of the form

$$\frac{Ax+B}{q(x)}.$$

To use this method, we need to be able to integrate each partial fraction. We know how to integrate terms of the form $A/(x-c)^n$ using the power rule (if $n > 1$) and logarithms (if $n = 1$). Next we see how to integrate terms of the form $(Ax+B)/q(x)$, where $q(x)$ is an unfactorable quadratic.

Trigonometric Substitutions

Section 7.1 showed how substitutions could be used to transform complex integrands. Now we see how substitution of $\sin\theta$ or $\tan\theta$ can be used for integrands involving square roots of quadratics or unfactorable quadratics.

Sine Substitutions

Substitutions involving $\sin\theta$ make use of the Pythagorean identity, $\cos^2\theta + \sin^2\theta = 1$, to simplify an integrand involving $\sqrt{a^2 - x^2}$.

Example 6

Find $\int \frac{1}{\sqrt{1-x^2}} dx$ using the substitution $x = \sin\theta$.

Solution

If $x = \sin\theta$, then $dx = \cos\theta d\theta$, and substitution gives

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta = \int \frac{\cos\theta}{\sqrt{\cos^2\theta}} d\theta .$$

Now either $\sqrt{\cos^2\theta} = \cos\theta$ or $\sqrt{\cos^2\theta} = -\cos\theta$ depending on the values taken by θ. If we choose $-\pi/2 \le \theta \le \pi/2$, then $\cos\theta \ge 0$, so $\sqrt{\cos^2\theta} = \cos\theta$. Then

$$\int \frac{\cos\theta}{\sqrt{\cos^2\theta}} d\theta = \int \frac{\cos\theta}{\cos\theta} d\theta = \int 1 d\theta = \theta + C = \arcsin x + C .$$

The last step uses the fact that $\theta = \arcsin x$ if $x = \sin\theta$ and $-\pi/2 \le \theta \le \pi/2$.

From now on, when we substitute $\sin\theta$, we assume that the interval $-\pi/2 \le \theta \le \pi/2$ has been chosen. Notice that the previous example is the case $a = 1$ of VI-28 in the table of integrals. The next example illustrates how to choose the substitution when $a \ne 1$.

Example 7

Use a trigonometric substitution to find $\int \frac{1}{\sqrt{4-x^2}} dx$.

Solution

This time we choose $x = 2\sin\theta$, with $-\pi/2 \le \theta \le \pi/2$, so that $4 - x^2$ becomes a perfect square:

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2\cos\theta .$$

Then $dx = 2\cos\theta d\theta$, so substitution gives

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{2\cos\theta} 2\cos\theta d\theta = \int 1 d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C .$$

The general rule for choosing a sine substitution is:

To simplify $\sqrt{a^2 - x^2}$, for constant a, try $x = a \sin\theta$, with $-\pi/2 \le \theta \le \pi/2$.

Notice $\sqrt{a^2 - x^2}$ is only defined on the interval $[-a, a]$. Assuming that the domain of the integrand is $[-a, a]$, the substitution $x = a \sin\theta$, with $-\pi/2 \le \theta \le \pi/2$, is valid for all x in the domain, because its range is $[-a, a]$ and it has an inverse $\theta = \arcsin(x/a)$ on $[-a, a]$.

Example 8

Find the area of the ellipse $4x^2 + y^2 = 9$.

Solution

Solving for y shows that $y = \sqrt{9 - 4x^2}$ gives the upper half of the ellipse. From Figure 7.1, we see that

$$\text{Area} = 4\int_0^{3/2} \sqrt{9 - 4x^2}\,dx\,.$$

To decide which trigonometric substitution to use, we write the integrand as

$$\sqrt{9 - 4x^2} = 2\sqrt{\frac{9}{4} - x^2} = 2\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,.$$

This suggests that we should choose $x = (3/2)\sin\theta$, so that $dx = (3/2)\cos\theta\,d\theta$ and

$$\sqrt{9 - 4x^2} = 2\sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 \sin^2\theta} = 2\left(\frac{3}{2}\right)\sqrt{1 - \sin^2\theta} = 3\cos\theta\,.$$

When $x = 0, \theta = 0$, and when $x = 3/2, \theta = \pi/2$, so

$$4\int_0^{3/2} \sqrt{9 - 4x^2}\,dx = 4\int_0^{\pi/2} 3\cos\theta\left(\frac{3}{2}\right)\cos\theta\,d\theta = 18\int_0^{\pi/2} \cos^2\theta\,d\theta\,.$$

Using Example 6 or table of integrals IV-18, we find

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\cos\theta\,\sin\theta + \frac{1}{2}\theta + C\,.$$

So we have

$$\text{Area} = 4\int_0^{3/2} \sqrt{9 - 4x^2}\,dx = \frac{18}{2}(\cos\theta\,\sin\theta + \theta)\Big|_0^{\pi/2} = 9\left(0 + \frac{\pi}{2}\right) = \frac{9\pi}{2}\,.$$

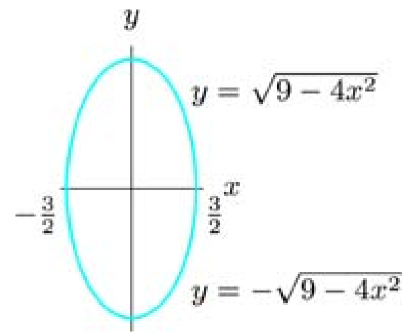

Figure 7.1: The ellipse $4x^2 + y^2 = 9$

Tangent Substitutions

Integrals involving $a^2 + x^2$ may be simplified by a substitution involving $\tan\theta$ and the trigonometric identities $\tan\theta = \sin\theta / \cos\theta$ and $\cos^2\theta + \sin^2\theta = 1$.

Example 9

Find $\int \frac{1}{x^2 + 9} dx$ using the substitution $x = 3\tan\theta$.

Solution

If $x = 3\tan\theta$, then $dx = (3 / \cos^2\theta)\, d\theta$, so

$$\int \frac{1}{x^2+9} dx = \int \left(\frac{1}{9\tan^2\theta + 9}\right)\left(\frac{3}{\cos^2\theta}\right) d\theta = \frac{1}{3}\int \frac{1}{\left(\frac{\sin^2\theta}{\cos^2\theta} + 1\right)\cos^2\theta} d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sin^2\theta + \cos^2\theta} d\theta = \frac{1}{3}\int 1\, d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

> To simplify $a^2 + x^2$ or $\sqrt{a^2 + x^2}$, for constant a, try $x = a\tan\theta$, with $-\pi/2 < \theta < \pi/2$.

Note that $a^2 + x^2$ and $\sqrt{a^2 + x^2}$ are defined on $(-\infty, \infty)$. Assuming that the domain of the integrand is $(-\infty, \infty)$, the substitution $x = a\tan\theta$, with $-\pi/2 < \theta < \pi/2$, is valid for all x in the domain, because its range is $(-\infty, \infty)$ and it has an inverse $\theta = \arctan(x / a)$ on $(-\infty, \infty)$.

Example 10

Use a tangent substitution to show that the following two integrals are equal:

$$\int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \frac{1}{\cos^3\theta}\,d\theta .$$

What area do these integrals represent? **Solution**

We put $x = \tan\theta$, with $-\pi/2 < \theta < \pi/2$, so that $dx = (1/\cos^2\theta)\,d\theta$, and

$$\sqrt{1+x^2} = \sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}} = \sqrt{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} = \frac{1}{\cos\theta} .$$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \pi/4$, so

$$\int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\cos^2\theta}\right)d\theta = \int_0^{\pi/4} \frac{1}{\cos^3\theta}\,d\theta .$$

The left-hand integral represents the area under the hyperbola $y^2 - x^2 = 1$ in Figure 7.2.

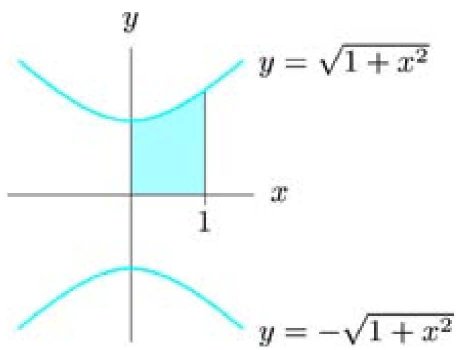

Figure 7.2: The hyperbola $y^2 - x^2 = 1$

Completing the Square to Use a Trigonometric Substitution

To make a trigonometric substitution, we may first need to complete the square.

Example 11

Find $\int \frac{3}{\sqrt{2x - x^2}} dx$.

Solution

To use a sine or tangent substitution, the expression under the square root sign should be in the form $a^2 + x^2$ or $a^2 - x^2$. Completing the square, we get

$$2x - x^2 = 1 - (x-1)^2 .$$

This suggests we substitute $x - 1 = \sin\theta$, or $x = \sin\theta + 1$. Then $dx = \cos\theta \, d\theta$, and

$$\int \frac{3}{\sqrt{2x - x^2}} dx = \int \frac{3}{\sqrt{1 - (x-1)^2}} dx = \int \frac{3}{\sqrt{1 - \sin^2\theta}} \cos\theta d\theta$$

$$= \int \frac{3}{\cos\theta} \cos\theta d\theta = \int 3 \, d\theta = 3\theta + C .$$

Since $x - 1 = \sin\theta$, we have $\theta = \arcsin(x - 1)$, so

$$\int \frac{3}{\sqrt{2x - x^2}} dx = 3 \arcsin(x-1) + C .$$

Example 12

Find $\int \frac{1}{x^2 + x + 1} dx$.

Solution

Completing the square, we get

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 .$$

This suggests we substitute $x + 1/2 = (\sqrt{3}/2)\tan\theta$, or $x = -1/2 + (\sqrt{3}/2)\tan\theta$. Then $dx = (\sqrt{3}/2)(1/\cos^2\theta)d\theta$, so

$$\int \frac{1}{x^2+x+1}dx = \int \left(\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \right) \left(\frac{\sqrt{3}}{2} \frac{1}{\cos^2\theta} \right) d\theta$$

$$= \frac{\sqrt{3}}{2} \int \left(\frac{1}{\frac{3}{4}\tan^2\theta + \frac{3}{4}} \right) \left(\frac{1}{\cos^2\theta} \right) d\theta = \frac{2}{\sqrt{3}} \int \frac{1}{(\tan^2\theta+1)\cos^2\theta} d\theta$$

$$= \frac{2}{\sqrt{3}} \int \frac{1}{\sin^2\theta+\cos^2\theta} d\theta = \frac{2}{\sqrt{3}} \int 1 d\theta = \frac{2}{\sqrt{3}}\theta + C.$$

Since $x + 1/2 = (\sqrt{3}/2)\tan\theta$, we have $\theta = \arctan((2/\sqrt{3})x + 1/\sqrt{3})$, so

$$\int \frac{1}{x^2+x+1}dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C.$$

Exercises and Problems for Section 7.4

Click here to open Student Solutions Manual: Ch 07 Section 04

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Exercises

Split the functions in Exercises 1–7 into partial fractions.

1. $\dfrac{20}{25-x^2}$

2. $\dfrac{x+1}{6x+x^2}$

3. $\dfrac{8}{y^3-4y}$

4. $\dfrac{2(1+s)}{s(s^2+3s+2)}$

5. $\dfrac{2}{s^4-1}$

6. $\dfrac{2y}{y^3-y^2+y-1}$

7. $\dfrac{1}{w^4-w^3}$

8. Integrate the function in Exercise 1 with respect to *x*.

9. Integrate the function in Exercise 2 with respect to *x*.

10. Integrate the function in Exercise 3 with respect to *y*.

11. Integrate the function in Exercise 4 with respect to *s*.

12. Integrate the function in Exercise 5 with respect to *s*.

13. Integrate the function in Exercise 6 with respect to *y*.

14. Integrate the function in Exercise 7 with respect to *w*.

In Exercises 15–19, evaluate the integral.

15. $\displaystyle\int \frac{3x^2-8x+1}{x^3-4x^2+x+6}dx$; use $\dfrac{A}{x-2}+\dfrac{B}{x+1}+\dfrac{C}{x-3}$.

16. $\displaystyle\int \frac{dx}{x^3-x^2}$; use $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}$.

17. $\displaystyle\int \frac{10x+2}{x^3-5x^2+x-5}dx$; use $\dfrac{A}{x-5}+\dfrac{Bx+C}{x^2+1}$.

18. $\displaystyle\int \frac{x^4+12x^3+15x^2+25x+11}{x^3+12x^2+11x}dx$; use division and $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+11}$.

19. $\displaystyle\int \frac{x^4+3x^3+2x^2+1}{x^2+3x+2}dx$; use division.

Use the given substitution in Exercises 20–22.

20. $\int \frac{1}{\sqrt{9-4x^2}} dx, x = \frac{3}{2}\sin t$

21. $\int \frac{1}{x^2+4x+5} dx, x = \tan t - 2$

22. $\int \frac{1}{\sqrt{4x-3-x^2}} dx, x = \sin t + 2$

23. Give a substitution (not necessarily trigonometric) which could be used to compute the following integrals:

(a). $\int \frac{x}{\sqrt{x^2+10}} dx$

(b). $\int \frac{1}{\sqrt{x^2+10}} dx$

Problems

Complete the square and give a substitution (not necessarily trigonometric) which could be used to compute the integrals in Problems 24–31.

24. $\int \frac{1}{x^2+2x+2} dx$

25. $\int \frac{1}{x^2+6x+25} dx$

26. $\int \frac{dy}{y^2+3y+3}$

27. $\int \frac{x+1}{x^2+2x+2} dx$

28. $\int \frac{4}{\sqrt{2z-z^2}} dz$

29. $\int \frac{z-1}{\sqrt{2z-z^2}} dz$

30. $\int (t+2)\sin(t^2+4t+7)\, dt$

31. $\int (2-\theta)\cos(\theta^2-4\theta)\, d\theta$

Calculate the integrals in Problems 32–49.

32. $\int \frac{1}{(x-5)(x-3)} dx$

33. $\int \frac{1}{(x+2)(x+3)} dx$

34. $\int \frac{1}{(x+7)(x-2)} dx$

35. $\int \frac{x}{x^2-3x+2} dx$

36. $\int \frac{dz}{z^2+z}$

37. $\int \frac{dx}{x^2+5x+4}$

38. $\int \frac{dP}{3P-3P^2}$

39. $\int \frac{3x+1}{x^2-3x+2} dx$

40. $\int \frac{y+2}{2y^2+3y+1} dy$

41. $\int \frac{x+1}{x^3+x}dx$

42. $\int \frac{x-2}{x^2+x^4}dx$

43. $\int \frac{x^2}{\sqrt{9-x^2}}dx$

44. $\int \frac{y^2}{25+y^2}dy$

45. $\int \frac{dt}{t^2\sqrt{1+t^2}}$

46. $\int \frac{dz}{(4-z^2)^{3/2}}$

47. $\int \frac{10}{(s+2)(s^2+1)}ds$

48. $\int \frac{1}{x^2+4x+13}dx$

49. $\int \frac{e^x dx}{(e^x-1)(e^x+2)}$

Find the exact area of the regions in Problems 50–55.

50. Bounded by $3x/((x-1)(x-4))$, $y=0$, $x=2$, $x=3$.

51. Bounded by $y=(3x^2+x)/((x^2+1)(x+1))$, $y=0$, $x=0$, $x=1$.

52. Bounded by $y=x^2/\sqrt{1-x^2}$, $y=0$, $x=0$, $x=1/2$.

53. Bounded by $y=x^3/\sqrt{4-x^2}$, $y=0$, $x=0$, $x=\sqrt{2}$.

54. Bounded by $y = 1 / \sqrt{x^2 + 9}, y = 0, x = 0, x = 3.$

55. Bounded by $y = 1 / (x\sqrt{x^2 + 9}), y = 0, x = \sqrt{3}, x = 3.$

Calculate the integrals in Problems 56–58 by partial fractions and using the indicated substitution. Show that the results you get are the same.

56. $\int \frac{dx}{1 - x^2}$; substitution $x = \sin\theta.$

57. $\int \frac{2x}{x^2 - 1} dx$; substitution $w = x^2 - 1.$

58. $\int \frac{3x^2 + 1}{x^3 + x} dx$; substitution $w = x^3 + x.$

59.

(a). Show $\int \frac{1}{\sin^2\theta} d\theta = -\frac{1}{\tan\theta} + C.$

(b). Calculate $\int \frac{dy}{y^2\sqrt{5 - y^2}}.$

60. Calculate the integral $\int \frac{1}{(x - a)(x - b)} dx$ for

(a). $a \neq b$

(b). $a = b$

61. Calculate the integral $\int \frac{x}{(x - a)(x - b)} dx$ for

(a). $a \neq b$

(b). $a = b$

62. Calculate the integral $\int \frac{1}{x^2 - a} dx$ for

(a). $a > 0$
(b). $a = 0$
(c). $a < 0$

63. A rumor is spread in a school. For $0 < a < 1$ and $b > 0$, the time *t* at which a fraction *p* of the school population has heard the rumor is given by

$$t(p) = \int_a^p \frac{b}{x(1-x)} dx \,.$$

(a). Evaluate the integral to find an explicit formula for $t(p)$. Write your answer so it has only one ln term.
(b). At time $t = 0$ one percent of the school population ($p = 0.01$) has heard the rumor. What is *a*?
(c). At time $t = 1$ half the school population ($p = 0.5$) has heard the rumor. What is *b*?
(d). At what time has 90% of the school population ($p = 0.9$) heard the rumor?

64. The Law of Mass Action tells us that the time, *T*, taken by a chemical to create a quantity x_0 of the product (in molecules) is given by

$$T = \int_0^{x_0} \frac{k dx}{(a-x)(b-x)}$$

where *a* and *b* are initial quantities of the two ingredients used to make the product, and *k* is a positive constant. Suppose $0 < a < b$.

(a). Find the time taken to make a quantity $x_0 = a / 2$ of the product.
(b). What happens to *T* as $x_0 \rightarrow a$?