7.5 Approximating Definite Integrals

The methods of the last few sections allow us to get exact answers for definite integrals in a variety of cases. However, many functions do not have elementary antiderivatives. To evaluate the definite integrals of such functions, we cannot use the Fundamental Theorem; we must use numerical methods.

We already know how to approximate a definite integral numerically using left- and right-hand Riemann sums. In the next two sections we introduce better methods for approximating definite integrals—better in the sense that they give more accurate results with less work than that required to find the left- and right-hand sums.

The Midpoint Rule

In the left- and right-hand Riemann sums, the heights of the rectangles are found using the left-hand or right-hand endpoints, respectively, of the subintervals. For the *midpoint rule*, we use the midpoint of each of the subintervals.

For example, in approximating $\int_1^2 f(x)\,dx$ by a Riemann sum with two subdivisions, we first divide the interval $1 \le x \le 2$ into two pieces. The midpoint of the first subinterval is 1.25 and the midpoint of the second is 1.75. The heights of the two rectangles are $f(1.25)$ and $f(1.75)$, respectively. (See Figure 7.3.) The Riemann sum is

$$f(1.25)0.5 + f(1.75)0.5\,.$$

Figure 7.3 shows that evaluating f at the midpoint of each subdivision usually gives a better approximation to the area under the curve than evaluating f at either end. For this particular f, it appears that each rectangle is partly above and partly below the graph on each subinterval. Furthermore, the area under the curve which is not under the rectangle appears to be nearly equal to the area under the rectangle which is above the curve. In fact, this new midpoint Riemann sum is generally a better approximation to the definite integral than the left- or right-hand sum with the same number of subdivisions, n.

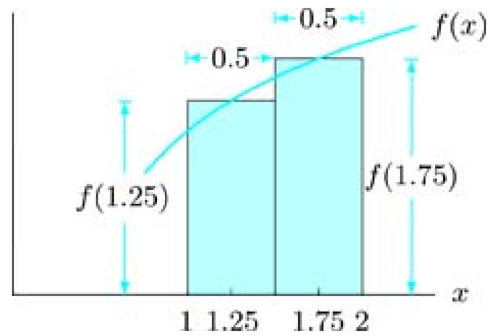


Figure 7.3: Midpoint rule with two subdivisions

So far, we have three ways of estimating an integral using a Riemann sum:

1. The **left rule** uses the left endpoint of each subinterval.
2. The **right rule** uses the right endpoint of each subinterval.
3. The **midpoint rule** uses the midpoint of each subinterval.

We write LEFT(n), RIGHT(n), and MID(n) to denote the results obtained by using these rules with n subdivisions.

Example 1

For $\int_1^2 \frac{1}{x} dx$, compute LEFT(2), RIGHT(2) and MID(2), and compare your answers with the exact value of the integral.

Solution

For $n = 2$ subdivisions of the interval [1, 2], we use $\Delta x = 0.5$. Then

$$\text{LEFT}(2) = f(1)(0.5) + f(1.5)(0.5) = \frac{1}{1}(0.5) + \frac{1}{1.5}(0.5) = 0.8333\ldots$$
$$\text{RIGHT}(2) = f(1.5)(0.5) + f(2)(0.5) = \frac{1}{1.5}(0.5) + \frac{1}{2}(0.5) = 0.5833\ldots$$
$$\text{MID}(2) = f(1.25)(0.5) + f(1.75)(0.5) = \frac{1}{1.25}(0.5) + \frac{1}{1.75}(0.5) = 0.6857\ldots$$

All three Riemann sums in this example are approximating

$$\int_1^2 \frac{1}{x} dx = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.6931\ldots.$$

With only two subdivisions, the left and right rules give quite poor approximations but the midpoint rule is already fairly close to the exact answer.

Figure 7.4(a) illustrates why the left and right rules are so inaccurate. Since $f(x) = 1/x$ is decreasing from 1 to 2, the left rule overestimates on each subdivision while the right rule underestimates. However, the midpoint rule approximates with rectangles on each subdivision that are each partly above and partly below the graph, so the errors tend to balance out. (See Figure 7.4(b).)

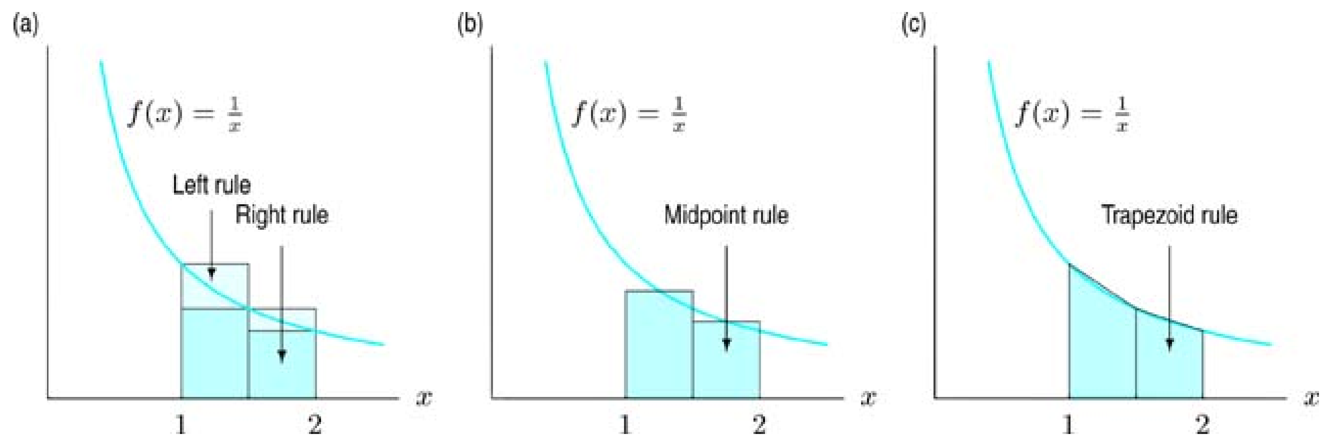


Figure 7.4: Left, right, midpoint, and trapezoid approximations to $\int_1^2 \frac{1}{x} dx$

The Trapezoid Rule

We have just seen how the midpoint rule can have the effect of balancing out the errors of the left and right rules. There is another way of balancing these errors: we average the results from the left and right rules. This approximation is called the *trapezoid rule*:

$$\text{TRAP}(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}.$$

The trapezoid rule averages the values of f at the left and right endpoints of each subinterval and multiplies by Δx. This is the same as approximating the area under the graph of f in each subinterval by a trapezoid (see Figure 7.5).

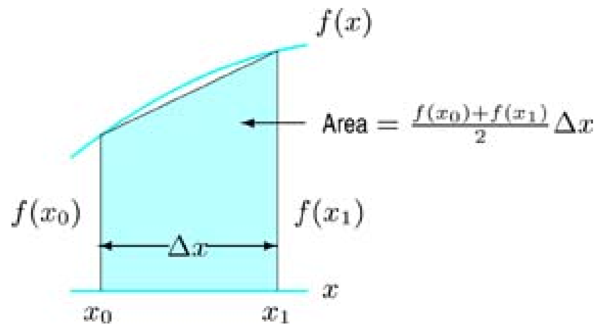


Figure 7.5: Area used in the trapezoid rule

Example 2

For $\int_1^2 \frac{1}{x} dx$, compare the trapezoid rule with two subdivisions with the left, right, and midpoint rules.

Solution

In the previous example we got $\text{LEFT}(2) = 0.8333\ldots$ and $\text{RIGHT}(2) = 0.5833\ldots$. The trapezoid rule is the average of these, so $\text{TRAP}(2) = 0.7083\ldots$. (See Figure 7.4(c).) The exact value of the integral is 0.6931 …, so the trapezoid rule is better than the left or right rules. The midpoint rule is still the best, however, since $\text{MID}(2) = 0.6857\ldots$.

Is the Approximation an Over- or Underestimate?

It is useful to know when a rule is producing an overestimate and when it is producing an underestimate. In Chapter 5 we saw that the following relationship holds.

> If f is increasing on $[a, b]$, then
>
> $$\text{LEFT}(n) \le \int_a^b f(x)dx \le \text{RIGHT}(n).$$
>
> If f is decreasing on $[a, b]$, then
>
> $$\text{RIGHT}(n) \le \int_a^b f(x)dx \le \text{LEFT}(n).$$

The Trapezoid Rule

If the graph of the function is concave down on [a, b], then each trapezoid lies below the graph and the trapezoid rule underestimates. If the graph is concave up on [a, b], the trapezoid rule overestimates. (See Figure 7.6.)

Figure 7.6: Error in the trapezoid rule

The Midpoint Rule

To understand the relationship between the midpoint rule and concavity, take a rectangle whose top intersects the curve at the midpoint of a subinterval. Draw a tangent to the curve at the midpoint; this gives a trapezoid. See Figure 7.7. (This is *not* the same trapezoid as in the trapezoid rule.) The midpoint rectangle and the new trapezoid have the same area, because the shaded triangles in Figure 7.7 are congruent. Hence, if the graph of the function is concave down, the midpoint rule overestimates; if the graph is concave up, the midpoint rule underestimates. (See Figure 7.8.)

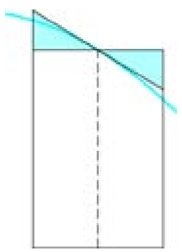


Figure 7.7: Midpoint rectangle and trapezoid with same area

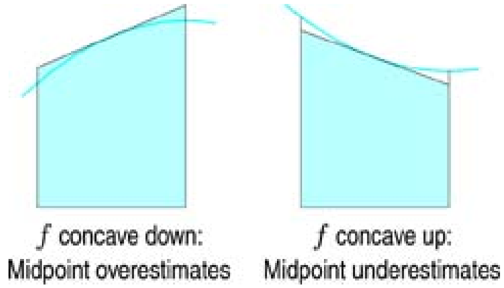


Figure 7.8: Error in the midpoint rule

If the graph of f is concave down on $[a, b]$, then

$$\text{TRAP}(n) \le \int_a^b f(x)dx \le \text{MID}(n).$$

If the graph of f is concave up on $[a, b]$, then

$$\text{MID}(n) \le \int_a^b f(x)dx \le \text{TRAP}(n).$$

Exercises and Problems for Section 7.5

Click here to open Student Solutions Manual: Ch 07 Section 05

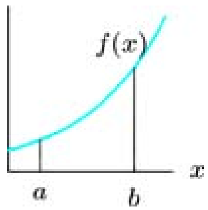
Click here to open Web Quiz Ch 07 Section 05

Exercises

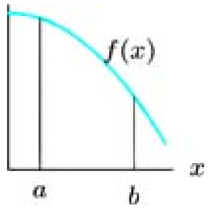
In Exercises 1–6, sketch the area given by the following approximations to $\int_a^b f(x)dx$. Identify each approximation as an overestimate or an underestimate.

(a). LEFT(2)
(b). RIGHT(2)
(c). TRAP(2)
(d). MID(2)

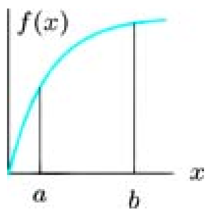
1.



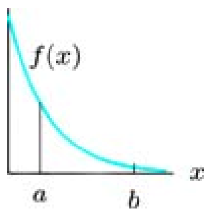
2.



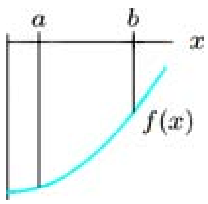
3.



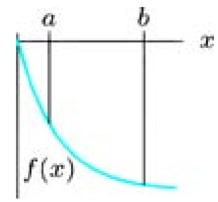
4.



5.



6.



7. Calculate the following approximations to $\int_0^6 x^2 dx$.

(a). LEFT(2)

(b). RIGHT(2)

(c). TRAP(2)

(d). MID(2)

8.

(a). Find LEFT(2) and RIGHT(2) for $\int_0^4 (x^2 + 1) dx$.

(b). Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

9.

(a). Find MID(2) and TRAP(2) for $\int_0^4 (x^2 + 1) dx$.

(b). Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

10. Calculate the following approximations to $\int_0^\pi \sin\theta \, d\theta$.

(a). LEFT(2)

(b). RIGHT(2)

(c). TRAP(2)

(d). MID(2)

Problems

11.

(a). Estimate $\int_0^1 1/(1+x^2)dx$ by subdividing the interval into eight parts using:

(i). The left Riemann sum.

(ii). The right Riemann sum.

(iii). The trapezoidal rule.

(b). Since the exact value of the integral is $\pi/4$, you can estimate the value of π using part(a). Explain why your first estimate is too large and your second estimate too small.

12. Using the table, estimate the total distance traveled from time $t=0$ to time $t=6$ using LEFT, RIGHT, and TRAP.

Time, *t*	0	1	2	3	4	5	6
Velocity, *v*	3	4	5	4	7	8	11

13. Using Figure 7.9, order the following approximations to the integral $\int_0^3 f(x)dx$ and its exact value from smallest to largest:

LEFT(*n*), RIGHT(*n*), MID(*n*), TRAP(*n*), Exact value.

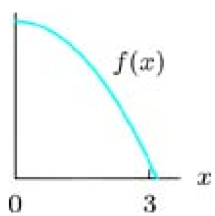


Figure 7.9

14. The results from the left, right, trapezoid, and midpoint rules used to approximate $\int_0^1 g(t)dt$, with the same number of subdivisions for each rule, are as follows:

0.601, 0.632, 0.633, 0.664.

(a). Using Figure 7.10, match each rule with its approximation.

(b). Between which two consecutive approximations does the true value of the integral lie?

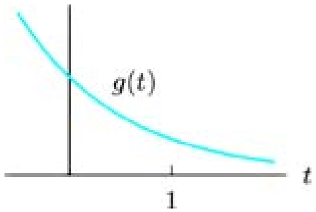
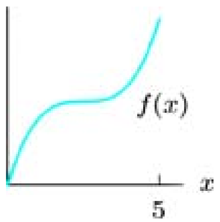


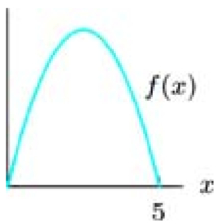
Figure 7.10

For the functions in Problems 15–18, pick which approximation — left, right, trapezoid, or midpoint — is guaranteed to give an overestimate for $\int_0^5 f(x)dx$, and which is guaranteed to give an underestimate. (There may be more than one.)

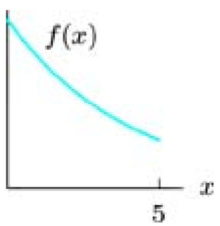
15.



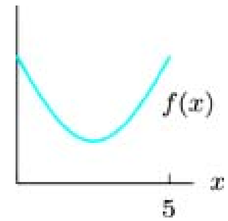
16.



17.



18.



19. Using a fixed number of subdivisions, we approximate the integrals of *f* and *g* on the interval in Figure 7.11.

(a). For which function, *f* or *g*, is LEFT more accurate? RIGHT? Explain.

(b). For which function, *f* or *g*, is TRAP more accurate? MID? Explain.

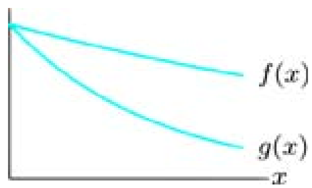


Figure 7.11

20.

(a). Values for $f(x)$ are in the table. Which of the four approximation methods in this section is most likely to give the best estimate of $\int_0^{12} f(x)dx$? Estimate the integral using this method.

(b). Assume $f(x)$ is continuous with no critical points or points of inflection on the interval $0 \le x \le 12$. Is the estimate found in part (a) an over- or underestimate? Explain.

x	0	3	6	9	12
$f(x)$	100	97	90	78	55

21.

(a). Find the exact value of $\int_0^{2\pi} \sin\theta\, d\theta$.

(b). Explain, using pictures, why the MID(1) and MID(2) approximations to this integral give the exact value.

(c). Does MID(3) give the exact value of this integral? How about MID(n)? Explain.

22.

(a). Show geometrically why $\int_0^1 \sqrt{2-x^2}dx = \frac{\pi}{4} + \frac{1}{2}$. [Hint: Break up the area under $y = \sqrt{2-x^2}$ from $x = 0$ to $x = 1$ into two pieces: a sector of a circle and a right triangle.]

(b). Approximate $\int_0^1 \sqrt{2-x^2}dx$ for $n = 5$ using the left, right, trapezoid, and midpoint rules. Compute the error in each case using the answer to part (a), and compare the errors.

23. The width, in feet, at various points along the fairway of a hole on a golf course is given in Figure 7.12. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway.

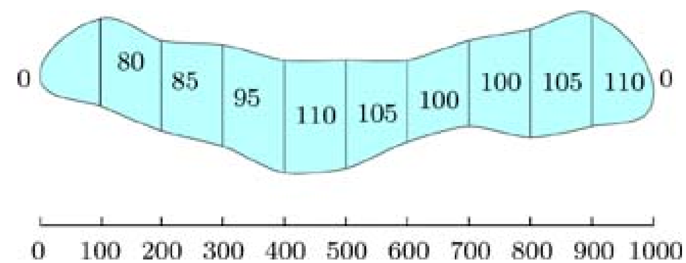


Figure 7.12

Problems 24–28 involve approximating $\int_a^b f(x)dx$.

24. Show $\text{RIGHT}(n) = \text{LEFT}(n) + f(b)\Delta x - f(a)\Delta x$.

25. Show $\text{TRAP}(n) = \text{LEFT}(n) + \frac{1}{2}(f(b) - f(a))\Delta x$.

26. Show $\text{LEFT}(2n) = \frac{1}{2}(\text{LEFT}(n) + \text{MID}(n))$.

27. Check that the equations in Problems 24 and 25 hold for $\int_1^2 (1/x)dx$ when $n = 10$.

28. Suppose that $a = 2$, $b = 5$, $f(2) = 13$, $f(5) = 21$ and that $\text{LEFT}(10) = 3.156$ and $\text{MID}(10) = 3.242$. Use Problems 24–26 to compute RIGHT(10), TRAP(10), LEFT(20), RIGHT(20), and TRAP(20).