7.6 Approximation Errors and Simpson's Rule

When we compute an approximation, we are always concerned about the error, namely the difference between the exact answer and the approximation. We usually do not know the exact error; if we did, we would also know the exact answer. Often the best we can get is an upper bound on the error and some idea of how much work is involved in making the error smaller. The study of numerical approximations is really the study of errors. The errors for some methods are much smaller than those for others. The errors for the midpoint and trapezoid rules are related to each other in a way that suggests an even better method, called Simpson's rule. We work with the example $\int_1^2 (1/x)dx$ because we know the exact value of this integral (ln 2) and we can investigate the behavior of the errors.

Error in Left and Right Rules

For any approximation, we take

$$\text{Error} = \text{Actual value} - \text{Approximate value}.$$

Let us see what happens to the error in the left and right rules as we increase *n*. We increase *n* each time by a factor of 5 starting at $n = 2$. The results are in Table 7.1. A positive error indicates that the Riemann sum is less than the exact value, ln 2. Notice that the errors for the left and right rules have opposite signs but are approximately equal in magnitude. (See Figure 7.13.) The best way to try to get the errors to cancel is to average the left and right rules; this average is the trapezoid rule. If we had not already thought of the trapezoid rule, we might have been led to invent it by this observation.

Table 7.1 *Errors for the left and right rule approximation to* $\int_1^2 \frac{1}{x} dx = \ln 2 \approx 0.6931471806$

n	Error in left rule	Error in right rule
2	–0.1402	0.1098
10	–0.0256	0.0244
50	–0.0050	0.0050
250	–0.0010	0.0010

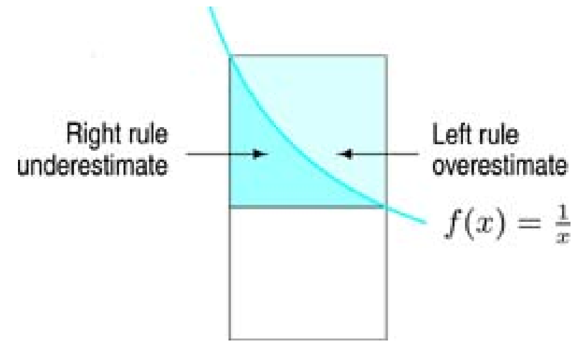

Figure 7.13: Errors in left and right sums

There is another pattern to the errors in Table 7.1. If we compute the *ratio* of the errors in Table 7.2, we see that the error[2] in both the left and right rules decreases by a factor of about 5 as *n* increases by a factor of 5.

Table 7.2 *Ratio of the errors as n increases for* $\int_1^2 \frac{1}{x} dx$

	Ratio of errors in left rule	Ratio of errors in right rule
Error(2)/Error(10)	5.47	4.51
Error(10)/Error(50)	5.10	4.90
Error(50)/Error(250)	5.02	4.98

There is nothing special about the number 5; the same holds for any factor. To get one extra digit of accuracy in any calculation, we must make the error $1/10$ as big, so we must increase *n* by a factor of 10. In fact, *for the left or right rules, each extra digit of accuracy requires about 10 times the work*. The calculator used to produce these tables took about half a second to compute the left rule approximation for $n = 50$, and this yields ln 2 to two digits. To get three correct digits, *n* would need to be around 500 and the time would be about 5 seconds. Four digits requires $n = 5000$ and 50 seconds. Ten digits requires $n = 5 \cdot 10^9$ and $5 \cdot 10^7$ seconds, which is more than a year! Clearly, the errors for the left and right rules do not decrease fast enough as *n* increases for practical use.

Error in Trapezoid and Midpoint Rules

Table 7.3 shows that the trapezoid and midpoint rules produce much better approximations to $\int_1^2 (1/x)dx$ than the left and right rules.

Table 7.3 *The errors for the trapezoid and midpoint rules for* $\int_1^2 \frac{1}{x}dx$

n	Error in trapezoid rule	Error in midpoint rule
2	–0.0152	0.0074
10	–0.00062	0.00031
50	–0.0000250	0.0000125
250	–0.0000010	0.0000005

Again there is a pattern to the errors. For each *n*, the midpoint rule is noticeably better than the trapezoid rule; the error for the midpoint rule, in absolute value, seems to be about half the error of the trapezoid rule. To see why, compare the shaded areas in Figure 7.14. Also, notice in Table 7.3 that the errors for the two rules have opposite signs; this is due to concavity.

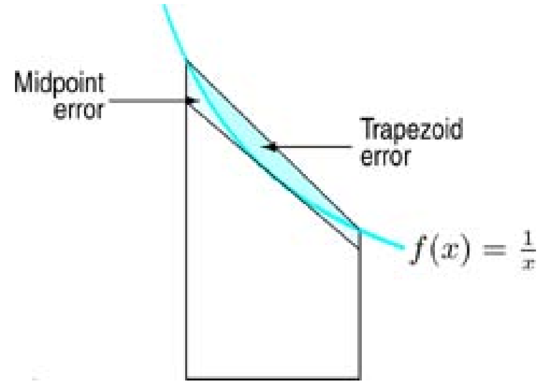

Figure 7.14: Errors in the midpoint and trapezoid rules

We are interested in how the errors behave as *n* increases. Table 7.4 gives the ratios of the errors for each rule. For each rule, we see that as *n* increases by a factor of 5, the error decreases by a factor of about $25 = 5^2$. In fact, it can be shown that this squaring relationship holds for any factor, so increasing *n* by a factor of 10 will decrease the error by a factor of about $100 = 10^2$. Reducing the error by a factor of 100 is equivalent to adding two more decimal places of accuracy to the result.

Table 7.4 *Ratios of the errors as n increases for* $\int_1^2 \frac{1}{x} dx$

	Ratio of errors in trapezoid rule	Ratio of errors in midpoint rule
Error(2)/Error(10)	24.33	23.84
Error(10)/Error(50)	24.97	24.95
Error(50)/Error(250)	25.00	25.00

In other words: *In the trapezoid or midpoint rules, each extra 2 digits of accuracy requires about 10 times the work.*

This result shows the advantage of the midpoint and trapezoid rules over the left and right rules: less additional work needs to be done to get another decimal place of accuracy. The calculator used to produce these tables again took about half a second to compute the midpoint rule for $\int_1^2 \frac{1}{x}dx$ with $n = 50$, and this gets 4 digits correct. Thus to get 6 digits would take $n = 500$ and 5 seconds, to get 8 digits would take 50 seconds, and to get 10 digits would take 500 seconds, or about 10 minutes. That is still not great, but it is certainly better than the 1 year required by the left or right rule.

Simpson's Rule

Still more improvement is possible. Observing that the trapezoid error has the opposite sign and about twice the magnitude of the midpoint error, we may guess that a weighted average of the two rules, with the midpoint rule weighted twice the trapezoid rule, will have a much smaller error. This approximation is called *Simpson's rule*[3]:

$$\text{SIMP}(n) = \frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3}.$$

Table 7.5 gives the errors for Simpson's rule. Notice how much smaller the errors are than the previous errors. Of course, it is a little unfair to compare Simpson's rule at $n = 50$, say, with the previous rules, because Simpson's rule must compute the value of *f* at both the midpoint and the endpoints of each subinterval and hence involves evaluating the function at twice as many points. We know by our previous analysis, however, that even if we did compute the other rules at $n = 100$ to compare with Simpson's rule at $n = 50$, the other errors would only decrease by a factor of 2 for the left and right rules and by a factor of 4 for the trapezoid and midpoint rules.

Table 7.5 *The errors for Simpson's rule and the ratios of the errors*

n	Error	Ratio
2	-0.0001067877	
		550.15
10	-0.0000001940	
		632.27
50	-0.0000000003	

We see in Table 7.5 that as *n* increases by a factor of 5, the errors decrease by a factor of about 600, or about 5^4. Again this behavior holds for any factor, so increasing *n* by a factor of 10 decreases the error by a factor of about 10^4. In other words: *In Simpson's rule, each extra 4 digits of accuracy requires about 10 times the work.*

This is a great improvement over either the midpoint or trapezoid rules, which only give two extra digits of accuracy when *n* is increased by a factor of 10. Simpson's rule is so efficient that we get 9 digits correct with $n = 50$ in about 1 second on our calculator. Doubling *n* will decrease the error by a factor of about $2^4 = 16$ and hence will give the tenth digit. The total time is 2 seconds, which is pretty good.

In general, Simpson's rule achieves a reasonable degree of accuracy when using relatively small values of *n*, and is a good choice for an all-purpose method for estimating definite integrals.

Analytical View of the Trapezoid and Simpson's Rules

Our approach to approximating $\int_a^b f(x)dx$ numerically has been empirical: try a method, see how the error behaves, and then try to improve it. We can also develop the various rules for numerical integration by making better and better approximations to the integrand, *f*. The left, right, and midpoint rules are all examples of approximating *f* by a constant (flat) function on each subinterval. The trapezoid rule is obtained by approximating *f* by a linear function on each subinterval. Simpson's rule can, in the same spirit, be obtained by approximating *f* by quadratic functions. The details are given in Problems 8 and 9.

How the Error Depends on the Integrand

Other factors besides the size of *n* affect the size of the error in each of the rules. Instead of looking at how the error behaves as we increase *n*, let's leave *n* fixed and imagine trying our approximation methods on different functions. We observe that the error in the left or right rule depends on how steeply the graph of *f* rises or falls. A steep curve makes the triangular regions missed by the left or right rectangles tall and hence large in area. This observation suggests that the error in the left or right rules depends on the size of the derivative of *f* (see Figure 7.15).

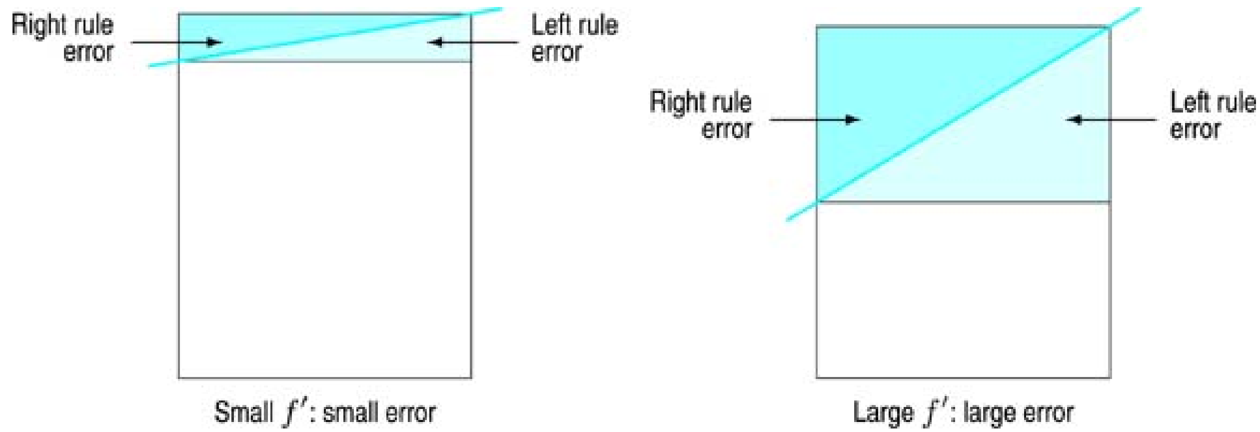

Figure 7.15: The error in the left and right rules depends on the steepness of the curve

From Figure 7.16 it appears that the errors in the trapezoid and midpoint rules depend on how much the curve is bent up or down. In other words, the concavity, and hence the size of the second derivative of *f*, has an effect on the errors of these two rules. Finally, it can be shown[4] that the error in Simpson's rule depends on the size of the *fourth* derivative of *f*, written $f^{(4)}$.

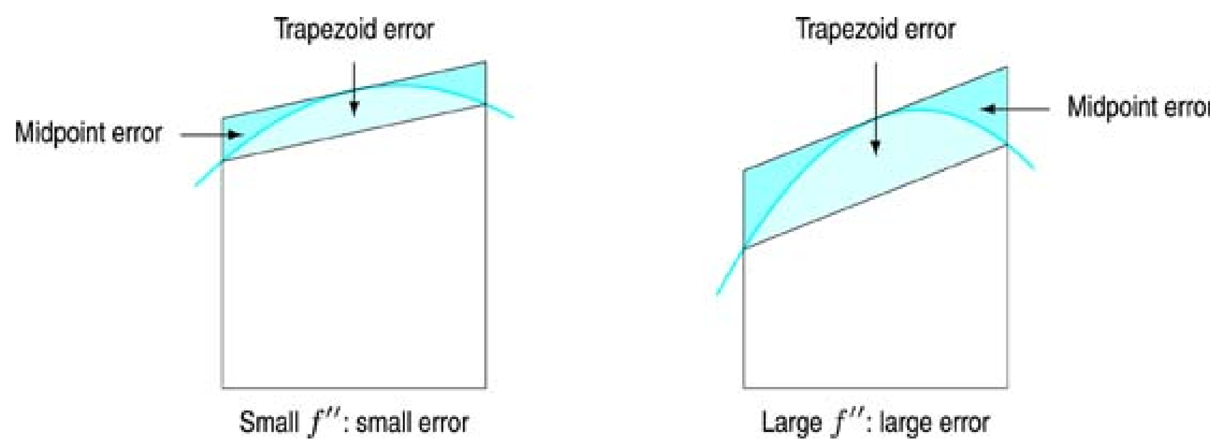

Figure 7.16: The error in the trapezoid and midpoint rules depends on how bent the curve is

Exercises and Problems for Section 7.6

Click here to open Student Solutions Manual: Ch 07 Section 06

Click here to open Web Quiz Ch 07 Section 06

Exercises

1. Estimate $\int_0^6 x^2 dx$ using SIMP(2).

2.

(a). Using the result of Problem 9, compute SIMP(2) for $\int_0^4 (x^2 + 1) dx$.

(b). Use the Fundamental Theorem of Calculus to find $\int_0^4 (x^2 + 1) dx$ exactly.

(c). What is the error in SIMP(2) for this integral?

Problems

3. In this problem you will investigate the behavior of the errors in the approximation of the integral

$$\int_1^2 \frac{1}{x} dx \approx 0.6931471806\ldots$$

(a). For $n = 2$, 4, 8, 16, 32, 64, 128 subdivisions, find the left and right approximations and the errors in each.

(b). What are the signs of the errors in the left and right approximations? How do the errors change if *n* is doubled?

(c). For the values of *n* in part (a), compute the midpoint and trapezoid approximations and the errors in each.

(d). What are the signs of the errors in the midpoint and trapezoid approximations? How do the errors change if *n* is doubled?

(e). For $n = 2$, 4, 8, 16, 32, compute Simpson's rule approximation and the error in each. How do these errors change as *n* doubles?

4.

(a). What is the exact value of $\int_0^2 (x^3 + 3x^2)dx$?

(b). Find $\text{SIMP}(n)$ for $n = 2$, 4, 100. What do you notice?

5.

(a). What is the exact value of $\int_0^4 e^x dx$?

(b). Find LEFT(2), RIGHT(2), TRAP(2), MID(2), and SIMP(2). Compute the error for each.

(c). Repeat part (b) with $n = 4$ (instead of $n = 2$).

(d). For each rule in part (b), as *n* goes from $n = 2$ to $n = 4$, does the error go down approximately as you would expect? Explain.

6. The approximation to a definite integral using $n = 10$ is 2.346; the exact value is 4.0. If the approximation was found using each of the following rules, use the same rule to estimate the integral with $n = 30$.

(a). LEFT

(b). TRAP

(c). SIMP

7.

(a). A certain computer takes two seconds to compute a certain definite integral accurate to 4 digits to the right of the decimal point, using the left rectangle rule. How long (in years) will it take to get 8 digits correct using the left rectangle rule? How about 12 digits? 20 digits?

(b). Repeat part (a) but this time assume that the trapezoidal rule is being used throughout.

Problems 8–9 show how Simpson's rule can be obtained by approximating the integrand, *f*, by quadratic functions.

8. Suppose that $a < b$ and that *m* is the midpoint $m = (a + b) / 2$. Let $h = b - a$. The purpose of this problem is to show that if *f* is a quadratic function, then

$$\int_a^b f(x)dx = \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right).$$

(a). Show that this equation holds for the functions $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$.

(b). Use part (a) and the properties of the integral to show that the equation holds for any quadratic function, $f(x) = Ax^2 + Bx + C$.

9. Consider the following method for approximating $\int_a^b f(x)dx$. Divide the interval [*a*, *b*] into *n* equal subintervals. On each subinterval approximate *f* by a quadratic function that agrees with *f* at both endpoints and at the midpoint of the subinterval.

(a). Explain why the integral of *f* on the subinterval $[x_i, x_{i+1}]$ is approximately equal to the expression

$$\frac{h}{3}\left(\frac{f(x_i)}{2} + 2f(m_i) + \frac{f(x_{i+1})}{2}\right),$$

where m_i is the midpoint of the subinterval, $m_i = (x_i + x_{i+1}) / 2$. (See Problem 8.)

(b). Show that if we add up these approximations for each subinterval, we get Simpson's rule:

$$\int_a^b f(x)dx \approx \frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3}.$$

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