7.7 Improper Integrals

Our original discussion of the definite integral $\int_a^b f(x)dx$ assumed that the interval $a \leq x \leq b$ was of finite length and that *f* was continuous. Integrals that arise in applications don't necessarily have these nice properties. In this section we investigate a class of integrals, called *improper* integrals, in which one limit of integration is infinite or the integrand is unbounded. As an example, to estimate the mass of the earth's atmosphere, we might calculate an integral which sums the mass of the air up to different heights. In order to represent the fact that the atmosphere does not end at a specific height, we let the upper limit of integration get larger and larger, or tend to infinity.

We will usually consider only improper integrals with positive integrands since they are the most common.

One Type of Improper Integral: When the Limit of Integration Is Infinite

Here is an example of an improper integral:

$$\int_1^\infty \frac{1}{x^2}dx .$$

To evaluate this integral, we first compute the definite integral $\int_1^b (1/x^2)dx$:

$$\int_1^b \frac{1}{x^2}dx = -x^{-1}\Big|_1^b = -\frac{1}{b} + \frac{1}{1} .$$

Now take the limit as $b \to \infty$. Since

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1,$$

we say that the improper integral $\int_1^\infty (1/x^2)dx$ *converges* to 1.

If we think in terms of areas, the integral $\int_1^\infty (1/x^2)dx$ represents the area under $f(x) = 1/x^2$ from $x = 1$ extending infinitely far to the right. (See Figure 7.17(a).) It may seem strange that this region has finite area. What our limit computations are saying is that

$$\text{When } b = 10: \quad \int_1^{10} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^{10} = -\frac{1}{10}+1 = 0.9$$

$$\text{When } b = 100: \quad \int_1^{100} \frac{1}{x^2}dx = -\frac{1}{100}+1 = 0.99$$

$$\text{When } b = 1000: \quad \int_1^{1000} \frac{1}{x^2}dx = -\frac{1}{1000}+1 = 0.999$$

and so on. In other words, as *b* gets larger and larger, the area between $x = 1$ and $x = b$ tends to 1. See Figure 7.17(b). Thus, it does make sense to declare that $\int_1^\infty (1/x^2)dx = 1$.

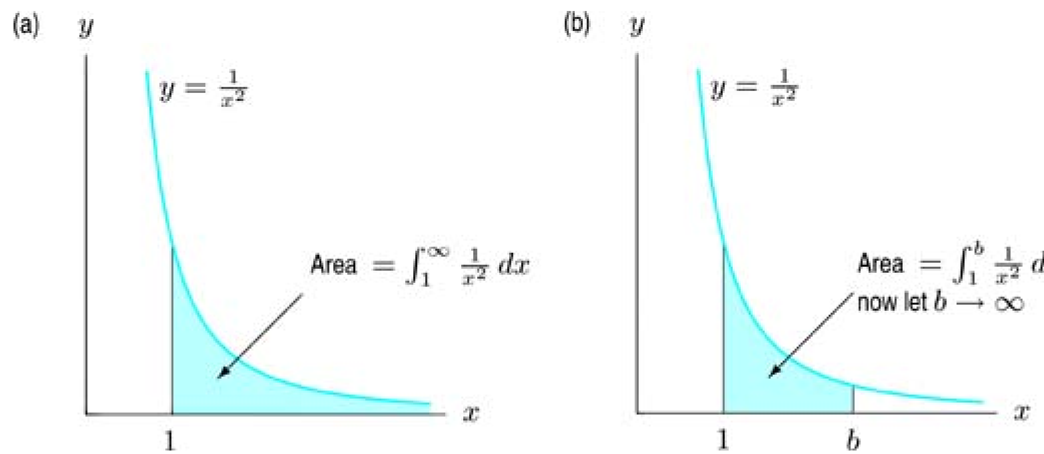

Figure 7.17: Area representation of improper integral

Of course, in another example, we might not get a finite limit as *b* gets larger and larger. In that case we say the improper integral *diverges*.

Suppose $f(x)$ is positive for $x \geq a$.

If $\lim_{b\to\infty}\int_a^b f(x)dx$ is a finite number, we say that $\int_a^\infty f(x)dx$ **converges** and define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx\,.$$

Otherwise, we say that $\int_a^\infty f(x)dx$ **diverges**. We define $\int_{-\infty}^{b} f(x)dx$ similarly.

Example 1

Does the improper integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ converge or diverge?

Solution

We consider

$$\int_1^b \frac{1}{\sqrt{x}}dx = \int_1^b x^{-1/2}dx = 2x^{1/2}\Big|_1^b = 2b^{1/2} - 2\,.$$

We see that $\int_1^b (1/\sqrt{x})dx$ grows without bound as $b \to \infty$. We have shown that the area under the curve in Figure 7.18 is not finite. Thus we say the integral $\int_1^\infty (1/\sqrt{x})dx$ *diverges*. We could also say $\int_1^\infty (1/\sqrt{x})dx = \infty$.

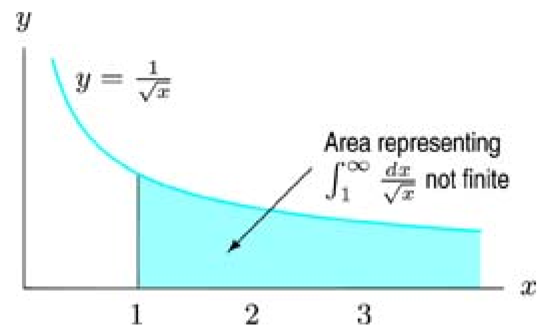

Figure 7.18: $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges

Notice that $f(x) \to 0$ as $x \to \infty$ does not guarantee convergence of $\int_a^\infty f(x)dx$.

What is the difference between the functions $1/x^2$ and $1/\sqrt{x}$ that makes the area under the graph of $1/x^2$ approach 1 as $x \to \infty$, whereas the area under $1/\sqrt{x}$ grows very large? Both functions approach 0 as *x* grows, so as *b* grows larger, smaller bits of area are being added to the definite integral. The difference between the functions is subtle: the values of the function $1/\sqrt{x}$ *don't shrink fast enough* for the integral to have a finite value. Of the two functions, $1/x^2$ drops to 0 much faster than $1/\sqrt{x}$, and this feature keeps the area under $1/x^2$ from growing beyond 1.

Example 2

Find $\int_0^\infty e^{-5x}dx$.

Solution

First we consider $\int_0^b e^{-5x}dx$:

$$\int_0^b e^{-5x}dx = -\frac{1}{5}e^{-5x}\Big|_0^b = -\frac{1}{5}e^{-5b} + \frac{1}{5}.$$

Since $e^{-5b} = \frac{1}{e^{5b}}$, this term tends to 0 as *b* approaches infinity, so $\int_0^\infty e^{-5x}dx$ converges. Its value is

$$\int_0^\infty e^{-5x}dx = \lim_{b\to\infty}\int_0^b e^{-5x}dx = \lim_{b\to\infty}\left(-\frac{1}{5}e^{-5b} + \frac{1}{5}\right) = 0 + \frac{1}{5} = \frac{1}{5}.$$

Since e^{5x} grows very rapidly, we expect that e^{-5x} will approach 0 rapidly. The fact that the area approaches $1/5$ instead of growing without bound is a consequence of the speed with which the integrand e^{-5x} approaches 0.

Example 3

Determine for which values of the exponent, *p*, the improper integral $\int_1^\infty \frac{1}{x^p}dx$ diverges.

Solution

For $p \neq 1$,

$$\int_1^b x^{-p}dx = \frac{1}{-p+1}x^{-p+1}\Big|_1^b = \left(\frac{1}{-p+1}b^{-p+1} - \frac{1}{-p+1}\right).$$

The important question is whether the exponent of *b* is positive or negative. If it is negative, then as *b* approaches infinity, b^{-p+1} approaches 0. If the exponent is positive, then b^{-p+1} grows without bound as *b* approaches infinity.

What happens if $p = 1$? In this case we get

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty} \ln b - \ln 1 .$$

Since ln *b* becomes arbitrarily large as *b* approaches infinity, the integral grows without bound. We conclude that $\int_1^\infty (1/x^p)dx$ diverges precisely when $p \leq 1$. For $p > 1$ the integral has the value

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty}\left(\frac{1}{-p+1}b^{-p+1} - \frac{1}{-p+1}\right) = -\left(\frac{1}{-p+1}\right) = \frac{1}{p-1} .$$

Application of Improper Integrals to Energy

The energy, *E*, required to separate two charged particles, originally a distance *a* apart, to a distance *b*, is given by the integral

$$E = \int_a^b \frac{kq_1q_2}{r^2}dr$$

where q_1 and q_2 are the magnitudes of the charges and *k* is a constant. If q_1 and q_2 are in coulombs, *a* and *b* are in meters, and *E* is in joules, the value of the constant *k* is $9 \cdot 10^9$.

Example 4

A hydrogen atom consists of a proton and an electron, with opposite charges of magnitude $1.6 \cdot 10^{-19}$ coulombs. Find the energy required to take a hydrogen atom apart (that is, to move the electron from its orbit to an infinite distance from the proton). Assume that the initial distance between the electron and the proton is the Bohr radius, $R_B = 5.3 \cdot 10^{-11}$ meter.

Solution

Since we are moving from an initial distance of R_B to a final distance of ∞, the energy is represented by the improper integral

$$E = \int_{R_B}^{\infty} k\frac{q_1 q_2}{r^2} dr = kq_1 q_2 \lim_{b \to \infty} \int_{R_B}^{b} \frac{1}{r^2} dr$$

$$= kq_1 q_2 \lim_{b \to \infty} -\frac{1}{r}\Big|_{R_B}^{b} = kq_1 q_2 \lim_{b \to \infty} \left(-\frac{1}{b} + \frac{1}{R_B}\right) = \frac{kq_1 q_2}{R_B}.$$

Substituting numerical values, we get

$$E = \frac{(9 \cdot 10^9)(1.6 \cdot 10^{-19})^2}{5.3 \cdot 10^{-11}} \approx 4.35 \cdot 10^{-18} \text{joules.}$$

This is about the amount of energy needed to lift a speck of dust 0.000000025 inch off the ground. (In other words, not much!)

What happens if the limits of integration are $-\infty$ and ∞? In this case, we break the integral at any point and write the original integral as a sum of two new improper integrals.

We can use any (finite) number *c* to define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx.$$

If *either* of the two new improper integrals diverges, we say the original integral diverges. Only if both of the new integrals have a finite value do we add the values to get a finite value for the original integral.

It is not hard to show that the preceding definition does not depend on the choice for *c*.

Another Type of Improper Integral: When the Integrand Becomes

Infinite

There is another way for an integral to be improper. The interval may be finite but the function may be unbounded near some points in the interval. For example, consider $\int_0^1 (1/\sqrt{x})dx$. Since the graph of $y = 1/\sqrt{x}$ has a vertical asymptote at $x = 0$, the region between the graph, the *x*-axis, and the lines $x = 0$ and $x = 1$ is unbounded. Instead of extending to infinity in the horizontal direction as in the previous improper integrals, this region extends to infinity in the vertical direction. See Figure 7.19(a). We handle this improper integral in a similar way as before: we compute $\int_a^1 (1/\sqrt{x})dx$ for values of *a* slightly larger than 0 and look at what happens as *a* approaches 0 from the positive side. (This is written as $a \to 0^+$.)

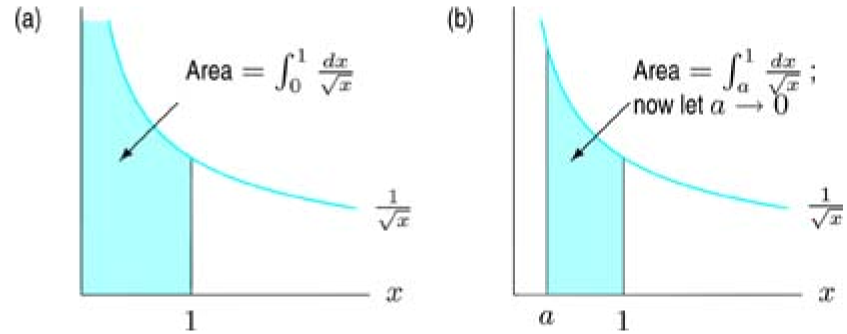

Figure 7.19: Area representation of improper integral

First we compute the integral:

$$\int_a^1 \frac{1}{\sqrt{x}}dx = 2x^{1/2}\Big|_a^1 = 2 - 2a^{1/2}.$$

Now we take the limit:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+}(2 - 2a^{1/2}) = 2.$$

Since the limit is finite, we say the improper integral converges, and that

$$\int_0^1 \frac{1}{\sqrt{x}}dx = 2.$$

Geometrically, what we have done is to calculate the finite area between $x = a$ and $x = 1$ and take the limit as *a* tends to 0 from the right. See Figure 7.19(b). Since the limit exists, the integral converges to 2. If the limit did not exist, we would say the improper integral diverges.

Example 5

Investigate the convergence of $\int_0^2 \frac{1}{(x-2)^2}dx$.

Solution

This is an improper integral since the integrand tends to infinity as *x* approaches 2, and is undefined at $x = 2$. Since the trouble is at the right endpoint, we replace the upper limit by *b*, and let *b* tend to 2 from the left. This is written $b \to 2^-$, with the "–" signifying that 2 is approached from below. See Figure 7.20.

$$\int_0^2 \frac{1}{(x-2)^2}dx = \lim_{b \to 2^-} \int_0^b \frac{1}{(x-2)^2}dx = \lim_{b \to 2^-} (-1)(x-2)^{-1}\Big|_0^b = \lim_{b \to 2^-} \left(-\frac{1}{(b-2)} - \frac{1}{2}\right).$$

Therefore, since $\lim_{b \to 2^-} \left(-\frac{1}{b-2}\right)$ does not exist, the integral diverges.

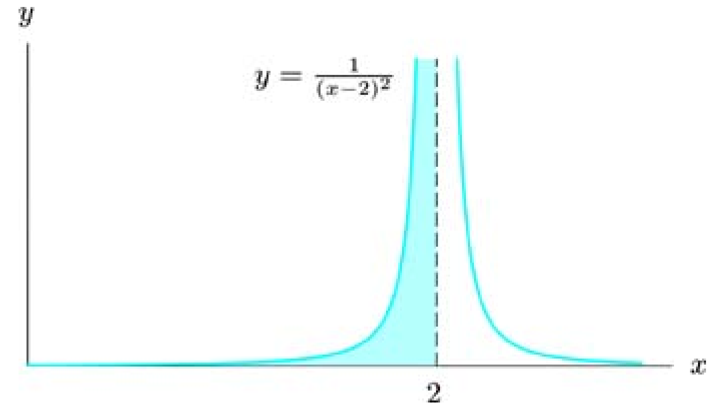

Figure 7.20: Shaded area represents $\int_0^2 \frac{1}{(x-2)^2}dx$

Suppose $f(x)$ is positive and continuous on $a \le x < b$ and tends to infinity as $x \to b$.

If $\lim_{c \to b^-} \int_a^c f(x)dx$ is a finite number, we say that $\int_a^b f(x)dx$ **converges** and define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx .$$

Otherwise, we say that $\int_a^b f(x)dx$ **diverges**.

When $f(x)$ tends to infinity as *x* approaches *a*, we define convergence in a similar way. In addition, an integral can be improper because the integrand tends to infinity *inside* the interval of integration rather than at an endpoint. In this case, we break the given integral into two (or more) improper integrals so that the integrand tends to infinity only at endpoints.

Suppose that $f(x)$ is positive and continuous on $[a, b]$ except at the point *c*. If $f(x)$ tends to infinity as $x \to c$, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

If *either* of the two new improper integrals diverges, we say the original integral diverges. Only if *both* of the new integrals have a finite value do we add the values to get a finite value for the original integral.

Example 6

Investigate the convergence of $\int_{-1}^{2} \frac{1}{x^4}dx$ . .

Solution

See the graph in Figure 7.21. The trouble spot is $x = 0$, rather than $x = -1$ or $x = 2$. To handle this situation, we break the given improper integral into two other improper integrals each of which have $x = 0$ as one of the endpoints:

$$\int_{-1}^{2} \frac{1}{x^4}dx = \int_{-1}^{0} \frac{1}{x^4}dx + \int_{0}^{2} \frac{1}{x^4}dx .$$

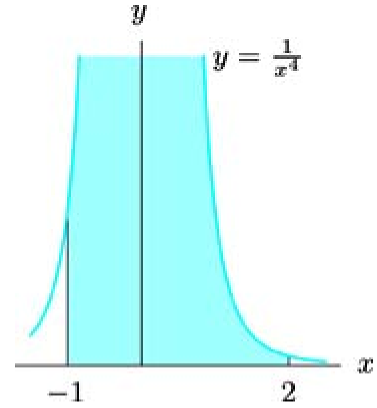

Figure 7.21: Shaded area represents $\int_{-1}^{2} \frac{1}{x^4} dx$

We can now use the previous technique to evaluate the new integrals, if they converge. Since

$$\int_0^2 \frac{1}{x^4} dx = \lim_{a \to 0^+} -\frac{1}{3}x^{-3}\Big|_a^2 = \lim_{a \to 0^+} \left(-\frac{1}{3}\right)\left(\frac{1}{8} - \frac{1}{a^3}\right)$$

the integral $\int_0^2 (1/x^4) dx$ diverges. Thus, the original integral diverges. A similar computation shows that $\int_{-1}^0 (1/x^4) dx$ also diverges.

It is easy to miss an improper integral when the integrand tends to infinity inside the interval. For example, it is fundamentally incorrect to say that $\int_{-1}^2 (1/x^4) dx = -\frac{1}{3}x^{-3}\Big|_{-1}^2 = -\frac{1}{24} - \frac{1}{3} = -\frac{3}{8}$.

Example 7

Find $\int_0^6 \frac{1}{(x-4)^{2/3}} dx$.

Solution

Figure 7.22 shows that the trouble spot is at $x = 4$, so we break the integral at $x = 4$ and consider the separate parts.

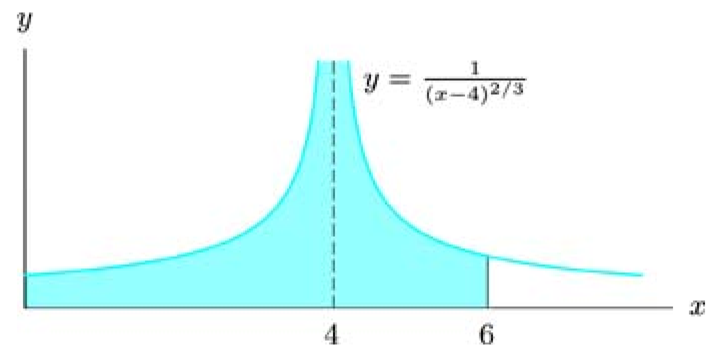

Figure 7.22: Shaded area represents $\int_0^6 \frac{1}{(x-4)^{2/3}} dx$

We have

$$\int_0^4 \frac{1}{(x-4)^{2/3}} dx = \lim_{b \to 4^-} 3(x-4)^{1/3} \Big|_0^b = \lim_{b \to 4^-} \left(3(b-4)^{1/3} - 3(-4)^{1/3}\right) = 3(4)^{1/3}.$$

Similarly,

$$\int_4^6 \frac{1}{(x-4)^{2/3}} dx = \lim_{a \to 4^+} 3(x-4)^{1/3} \Big|_a^6 = \lim_{a \to 4^+} \left(3 \cdot 2^{1/3} - 3(a-4)^{1/3}\right) = 3(2)^{1/3}.$$

Since both of these integrals converge, the original integral converges:

$$\int_0^6 \frac{1}{(x-4)^{2/3}} dx = 3(4)^{1/3} + 3(2)^{1/3} \approx 8.54.$$

Finally, there is a question of what to do when an integral is improper at both endpoints. In this case, we just break the integral at any interior point of the interval. The original integral diverges if either or both of the new integrals diverge.

Example 8

Investigate the convergence of $\int_0^\infty \frac{1}{x^2} dx$.

Solution

This integral is improper both because the upper limit is ∞ and because the function is undefined at $x = 0$. We break the integral into two parts at, say, $x = 1$. We know by Example 3 that $\int_1^\infty (1/x^2) dx$ has a finite value. However, the other part, $\int_0^1 (1/x^2) dx$, diverges since:

$$\int_0^1 \frac{1}{x^2} dx = \lim_{a \to 0^+} -x^{-1}\Big|_a^1 = \lim_{a \to 0^+} \left(\frac{1}{a} - 1\right).$$

Therefore $\int_0^\infty \frac{1}{x^2} dx$ diverges as well.

Exercises and Problems for Section 7.7

Click here to open Student Solutions Manual: Ch 07 Section 07

Click here to open Web Quiz Ch 07 Section 07

Exercises

1. Shade the area represented by:

(a). $\int_1^\infty (1/x^2) dx$

(b). $\int_0^1 (1/\sqrt{x}) dx$

2. Evaluate the improper integral $\int_0^\infty e^{-0.4x} dx$ and sketch the area it represents.

3.

(a). Use a calculator or computer to estimate $\int_0^b xe^{-x}dx$ for $b = 5, 10, 20$.

(b). Use your answers to part (a) to estimate the value of $\int_0^{\infty} xe^{-x}dx$, assuming it is finite.

4.

(a). Sketch the the area represented by the improper integral $\int_{-\infty}^{\infty} e^{-x^2}dx$.

(b). Use a calculator or computer to estimate $\int_{-a}^{a} e^{-x^2}dx$ for $a = 1, 2, 3, 4, 5$.

(c). Use the answers to part (b) to estimate the value of $\int_{-\infty}^{\infty} e^{-x^2}dx$, assuming it is finite.

Calculate the integrals in Exercises 5–32, if they converge.

5. $\int_1^{\infty} \frac{1}{5x+2}dx$

6. $\int_1^{\infty} \frac{1}{(x+2)^2}dx$

7. $\int_0^{\infty} xe^{-x^2}dx$

8. $\int_1^{\infty} e^{-2x}dx$

9. $\int_0^{\infty} \frac{x}{e^x}dx$

10. $\int_1^{\infty} \frac{x}{4+x^2}dx$

11. $\int_{-\infty}^{0} \frac{e^x}{1+e^x}dx$

12. $\int_{-\infty}^{\infty} \frac{dz}{z^2+25}$

13. $\int_0^4 \frac{dx}{\sqrt{16-x^2}}$

14. $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}dx$

15. $\int_{-1}^{1} \frac{1}{v}dv$

16. $\int_0^1 \frac{x^4+1}{x}dx$

17. $\int_1^{\infty} \frac{1}{x^2+1}dx$

18. $\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}dx$

19. $\int_0^4 \frac{1}{u^2-16}du$

20. $\int_1^{\infty} \frac{y}{y^4+1}dy$

21. $\int_2^{\infty} \frac{dx}{x \ln x}$

22. $\int_0^1 \frac{\ln x}{x}dx$

23. $\int_{16}^{20} \frac{1}{y^2-16}dy$

24. $\int_1^2 \frac{dx}{x \ln x}$

25. $\int_0^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx$

26. $\int_3^{\infty} \frac{dx}{x(\ln x)^2}$

27. $\int_0^{2} \frac{1}{\sqrt{4-x^2}} dx$

28. $\int_4^{\infty} \frac{dx}{(x-1)^2}$

29. $\int_4^{\infty} \frac{dx}{x^2-1}$

30. $\int_7^{\infty} \frac{dy}{\sqrt{y-5}}$

31. $\int_{-3}^{3} \frac{y\, dy}{\sqrt{9-y^2}}$

32. $\int_3^{6} \frac{d\theta}{(4-\theta)^2}$

Problems

33. Find the area under the curve $y = xe^{-x}$ for $x \geq 0$.

34. Find the area under the curve $y = 1 / \cos^2 t$ between $t = 0$ and $t = \pi / 2$.

For what values of *p* do the integrals in Problems 35–36 converge or diverge? What is the value of the integral when it converges?

35. $\int_e^{\infty} x^p \ln x\, dx$

36. $\int_0^{e} x^p \ln x\, dx$

37. For $\alpha > 0$, calculate

(a). $\int_0^\infty \frac{e^{-y/\alpha}}{\alpha} dy$

(b). $\int_0^\infty \frac{ye^{-y/\alpha}}{\alpha} dy$

(c). $\int_0^\infty \frac{y^2 e^{-y/\alpha}}{\alpha} dy$

38. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, calculate the exact value of

$$\int_{-\infty}^{\infty} e^{-(x-a)^2/b} dx .$$

The k^{th} moment, m_k of the normal distribution is defined by

$$m_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-x^2/2} dx .$$

In Problems 39–42, use the fact that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ to calculate the moments. Assume all the integrals converge.

39. m_1

40. m_2

41. m_3

42. m_4

43. The gamma function is defined for all $x > 0$ by the rule

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt .$$

(a). Find $\Gamma(1)$ and $\Gamma(2)$.

(b). Integrate by parts with respect to *t* to show that, for positive *n*,

$$\Gamma(n+1) = n\Gamma(n) .$$

(c). Find a simple expression for $\Gamma(n)$ for positive integers *n*.

44. The rate, *r*, at which people get sick during an epidemic of the flu can be approximated by $r = 1000te^{-0.5t}$, where *r* is measured in people/day and *t* is measured in days since the start of the epidemic.

(a). Sketch a graph of *r* as a function of *t*.

(b). When are people getting sick fastest?

(c). How many people get sick altogether?

45. Find the energy required to separate opposite electric charges of magnitude 1 coulomb. The charges are initially 1 meter apart and one is moved infinitely far from the other.