7.8 Comparison of Improper Integrals

Making Comparisons

Sometimes it is difficult to find the exact value of an improper integral by antidifferentiation, but it may be possible to determine whether an integral converges or diverges. The key is to *compare* the given integral to one whose behavior we already know. Let's look at an example.

Example 1

Determine whether $\int_1^{\infty} \frac{1}{\sqrt{x^3+5}}dx$ converges.

Solution

First, let's see what this integrand does as $x \to \infty$. For large *x*, the 5 becomes insignificant compared with the x^3, so

$$\frac{1}{\sqrt{x^3+5}} \approx \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}.$$

Since

$$\int_1^{\infty} \frac{1}{\sqrt{x^3}}dx = \int_1^{\infty} \frac{1}{x^{3/2}}dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty} \left. -2x^{-1/2}\right|_1^b = \lim_{b\to\infty}\left(2-2b^{-1/2}\right) = 2,$$

the integral $\int_1^{\infty}(1/x^{3/2})dx$ converges. So we expect our integral to converge as well.

In order to confirm this, we observe that for $0 \le x^3 \le x^3+5$, we have

$$\frac{1}{\sqrt{x^3+5}} \le \frac{1}{\sqrt{x^3}}.$$

and so for $b \ge 1$,

$$\int_1^b \frac{1}{\sqrt{x^3+5}}dx \le \int_1^b \frac{1}{\sqrt{x^3}}dx.$$

(See Figure 7.23.) Since $\int_1^b (1/\sqrt{x^3+5})dx$ increases as *b* approaches infinity but is always smaller than $\int_1^b (1/x^{3/2})dx < \int_1^{\infty}(1/x^{3/2})dx = 2$, we know $\int_1^{\infty}(1/\sqrt{x^3+5})\,dx$ must have a finite value less than 2. Thus,

$$\int_1^{\infty}\frac{dx}{\sqrt{x^3+5}} \text{ converges to a value less than } 2.$$

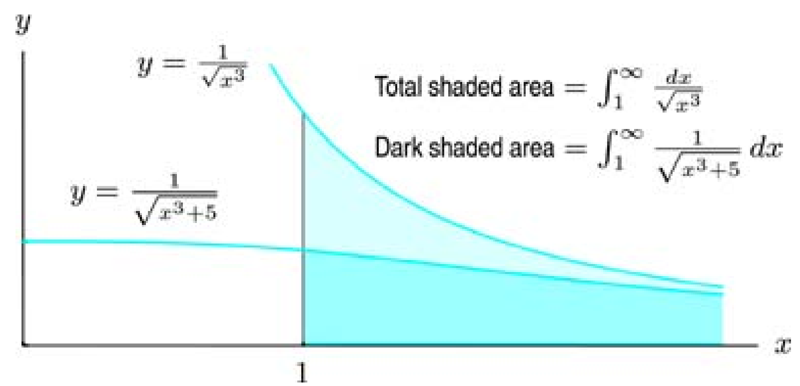

Figure 7.23: Graph showing $\int_1^{\infty}\frac{1}{\sqrt{x^3+5}}\,dx \le \int_1^{\infty}\frac{dx}{\sqrt{x^3}}$

A little more work is required to estimate the value of a convergent improper integral.

Example 2

Estimate the value of $\int_1^{\infty} \frac{dx}{\sqrt{x^3+5}}\,dx$ with an error of less than 0.01 using the approximation

$$\int_1^{\infty} \frac{1}{\sqrt{x^3+5}}\,dx \approx \int_1^{b} \frac{1}{\sqrt{x^3+5}}\,dx\,.$$

Solution

We must figure out how large a value of b to take. Since

$$\int_1^{\infty} \frac{1}{\sqrt{x^3+5}}\,dx = \int_1^{b} \frac{1}{\sqrt{x^3+5}}\,dx + \int_b^{\infty} \frac{1}{\sqrt{x^3+5}}\,dx,$$

we find b such that the tail of the integral satisfies the inequality

$$\left|\int_b^{\infty} \frac{1}{\sqrt{x^3+5}}dx\right| < 0.01\,.$$

From the solution of Example 1, we have

$$0 < \int_b^{\infty} \frac{1}{\sqrt{x^3+5}}dx < \int_b^{\infty} \frac{1}{\sqrt{x^3}}dx = \frac{2}{\sqrt{b}}\,.$$

We choose b such that $2 / \sqrt{b} < 0.01$, which means that $b > 40{,}000$. With an error of less than 0.01, we have

$$\int_1^{\infty} \frac{1}{\sqrt{x^3+5}}dx \approx \int_1^{50{,}000} \frac{1}{\sqrt{x^3+5}}dx = 1.699\,.$$

Notice that we first looked at the behavior of the integrand as $x \to \infty$. This is useful because the convergence or divergence of the integral is determined by what happens as $x \to \infty$.

The Comparison Test for $\int_a^{\infty} f(x)\,dx$

Assume $f(x)$ is positive. Making a comparison involves two stages:

1. Guess, by looking at the behavior of the integrand for large *x*, whether the integral converges or not. (This is the "behaves like" principle.)
2. Confirm the guess by comparison:
   - If $0 \le f(x) \le g(x)$ and $\int_a^{\infty} g(x)\,dx$ converges, then $\int_a^{\infty} f(x)\,dx$ converges.
   - If $0 \le g(x) \le f(x)$ and $\int_a^{\infty} g(x)\,dx$ diverges, then $\int_a^{\infty} f(x)\,dx$ diverges.

Example 3

Decide whether $\int_4^{\infty} \frac{dt}{(\ln t) - 1}$ converges or diverges.

Solution

Since ln *t* grows without bound as $t \to \infty$, the −1 is eventually going to be insignificant in comparison to ln *t*. Thus, as far as convergence is concerned,

$$\int_4^{\infty} \frac{1}{(\ln t) - 1}\,dt \quad \text{behaves like} \quad \int_4^{\infty} \frac{1}{\ln t}\,dt\,.$$

Does $\int_4^{\infty} (1/\ln t)\,dt$ converge or diverge? Since ln *t* grows very slowly, $1/\ln t$ goes to zero very slowly, and so the integral probably does not converge. We know that $(\ln t) - 1 < \ln t < t$ for all positive *t*. So, provided $t > e$, we take reciprocals:

$$\frac{1}{(\ln t) - 1} > \frac{1}{\ln t} > \frac{1}{t}\,.$$

Since $\int_4^{\infty} (1/t)\,dt$ diverges, we conclude that

$$\int_4^{\infty} \frac{1}{(\ln t) - 1}\,dt \text{ diverges.}$$

How Do We Know What To Compare With?

In Examples 1 and 3, we investigated the convergence of an integral by comparing it with an easier integral. How did we pick the easier integral? This is a matter of trial and error, guided by any information we get by looking at the original integrand as $x \to \infty$. We want the comparison integrand to be easy and, in particular, to have a simple antiderivative.

Useful Integrals for Comparison

- $\int_1^{\infty} \frac{1}{x^p}dx$ converges for $p > 1$ and diverges for $p \leq 1$.
- $\int_0^{1} \frac{1}{x^p}dx$ converges for $p < 1$ and diverges for $p \geq 1$.
- $\int_0^{\infty} e^{-ax}dx$ converges for $a > 0$.

Of course, we can use any function for comparison, provided we can determine its behavior.

Example 4

Investigate the convergence of $\int_1^{\infty} \frac{(\sin x)+3}{\sqrt{x}}\, dx$.

Solution

Since it looks difficult to find an antiderivative of this function, we try comparison. What happens to this integrand as $x \to \infty$? Since sin *x* oscillates between –1 and 1,

$$\frac{2}{\sqrt{x}} = \frac{-1+3}{\sqrt{x}} \leq \frac{(\sin x)+3}{\sqrt{x}} \leq \frac{1+3}{\sqrt{x}} = \frac{4}{\sqrt{x}},$$

the integrand oscillates between $2/\sqrt{x}$ and $4/\sqrt{x}$. (See Figure 7.24.)

What do $\int_1^{\infty} (2/\sqrt{x})dx$ and $\int_1^{\infty} (4/\sqrt{x})dx$ do? As far as convergence is concerned, they certainly do the same thing, and whatever that is, the original integral does it too. It is important to notice that $\sqrt{x}$ grows very slowly. This means that $1/\sqrt{x}$ gets small slowly, which means that convergence is unlikely. Since $\sqrt{x} = x^{1/2}$, the result in the preceding box (with $p = \frac{1}{2}$) tells us that $\int_1^{\infty} (1/\sqrt{x})dx$ diverges. So the comparison test tells us that the original integral diverges.

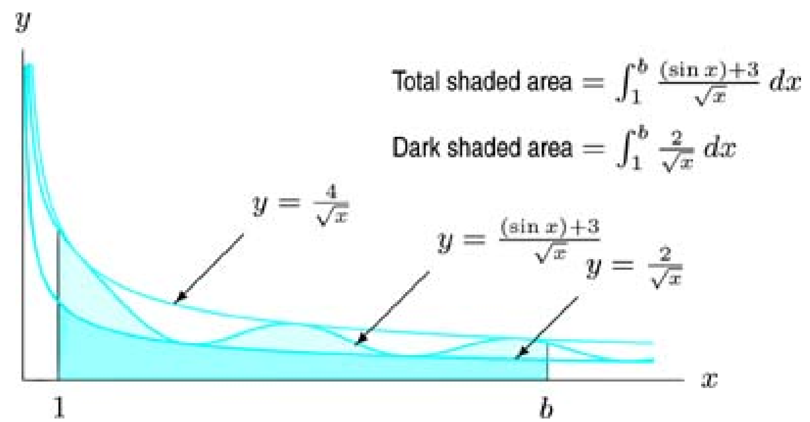

Figure 7.24: Graph showing $\int_1^b \frac{2}{\sqrt{x}}\, dx \leq \int_1^b \frac{(\sin x)+3}{\sqrt{x}} dx$, for $b \geq 1$

Notice that there are two possible comparisons we could have made in Example 4:

$$\frac{2}{\sqrt{x}} \leq \frac{(\sin x)+3}{\sqrt{x}} \quad \text{or} \quad \frac{(\sin x)+3}{\sqrt{x}} \leq \frac{4}{\sqrt{x}}.$$

Since both $\int_1^\infty (2/\sqrt{x})dx$ and $\int_1^\infty (4/\sqrt{x})dx$ diverge, only the first comparison is useful. Knowing that an integral is *smaller* than a divergent integral is of no help whatsoever!

The next example shows what to do if the comparison does not hold throughout the interval of integration.

Example 5

Show $\int_1^\infty e^{-x^2/2}\,dx$ converges to a finite value.

Solution

We know that $e^{-x^2/2}$ goes very rapidly to zero as $x \to \infty$, so we expect this integral to converge. Hence we look for some larger integrand which has a convergent integral. One possibility is $\int_1^\infty e^{-x}\,dx$, because e^{-x} has an elementary antiderivative and $\int_1^\infty e^{-x}\,dx$ converges. What is the relationship between $e^{-x^2/2}$ and e^{-x}? We know that for $x \geq 2$,

$$x \leq \frac{x^2}{2} \quad \text{so} \quad -\frac{x^2}{2} \leq -x,$$

and so, for $x \geq 2$

$$e^{-x^2/2} \leq e^{-x}.$$

Since this inequality holds only for $x \geq 2$, we split the interval of integration into two pieces:

$$\int_1^\infty e^{-x^2/2}dx = \int_1^2 e^{-x^2/2}dx + \int_2^\infty e^{-x^2/2}dx.$$

Now $\int_1^2 e^{-x^2/2}dx$ is finite (it is not improper) and $\int_2^\infty e^{-x^2/2}dx$ is finite by comparison with $\int_2^\infty e^{-x}dx$. Therefore, $\int_1^\infty e^{-x^2/2}dx$ is the sum of two finite pieces and therefore must be finite.

Exercises and Problems for Section 7.8

Click here to open Student Solutions Manual: Ch 07 Section 08

Click here to open Web Quiz Ch 07 Section 08

Exercises

In Exercises 1–9, the behavior of rational and exponential functions as $x \to \infty$ to predict whether the integrals converge or diverge.

1. $\int_1^\infty \frac{x^2}{x^4+1} dx$

2. $\int_2^\infty \frac{x^3}{x^4-1} dx$

3. $\int_1^\infty \frac{x^2+1}{x^3+3x+2} dx$

4. $\int_1^\infty \frac{1}{x^2+5x+1} dx$

5. $\int_1^\infty \frac{x}{x^2+2x+4} dx$

6. $\int_1^\infty \frac{x^2-6x+1}{x^2+4} dx$

7. $\int_1^\infty \frac{5x+2}{x^4+8x^2+4} dx$

8. $\int_1^\infty \frac{1}{e^{5t}+2} dt$

9. $\int_1^\infty \frac{x^2+4}{x^4+3x^2+11} dx$

In Exercises 10–25 decide if the improper integral converges or diverges. Explain your reasoning.

10. $\int_{50}^{\infty} \frac{dz}{z^3}$

11. $\int_{1}^{\infty} \frac{dx}{1+x}$

12. $\int_{1}^{\infty} \frac{dx}{x^3+1}$

13. $\int_{5}^{8} \frac{6}{\sqrt{t-5}}\, dt$

14. $\int_{0}^{1} \frac{1}{x^{19/20}}\, dx$

15. $\int_{-1}^{5} \frac{dt}{(t+1)^2}$

16. $\int_{-\infty}^{\infty} \frac{du}{1+u^2}$

17. $\int_{1}^{\infty} \frac{du}{u+u^2}$

18. $\int_{1}^{\infty} \frac{d\theta}{\sqrt{\theta^2+1}}$

19. $\int_{2}^{\infty} \frac{d\theta}{\sqrt{\theta^3+1}}$

20. $\int_{0}^{1} \frac{d\theta}{\sqrt{\theta^3+\theta}}$

21. $\int_{0}^{\infty} \frac{dy}{1+e^y}$

22. $\displaystyle\int_1^\infty \frac{2+\cos\phi}{\phi^2}d\phi$

23. $\displaystyle\int_0^\infty \frac{dz}{e^z+2^z}$

24. $\displaystyle\int_0^\pi \frac{2-\sin\phi}{\phi^2}d\phi$

25. $\displaystyle\int_4^\infty \frac{3+\sin\alpha}{\alpha}d\alpha$

Estimate the values of the integrals in Exercises 26–27 to two decimal places by integrating the functions on your calculator or computer for large values of the upper limit of integration.

26. $\displaystyle\int_1^\infty e^{-x^2}\,dx$

27. $\displaystyle\int_0^\infty e^{-x^2}\cos^2 x\,dx$

Problems

28. The graphs of $y = 1/x$, $y = 1/x^2$ and the functions $f(x)$, $g(x)$, $h(x)$, and $k(x)$ are shown in Figure 7.25.

(a). Is the area between $y = 1/x$ and $y = 1/x^2$ on the interval from $x = 1$ to ∞ finite or infinite? Explain.

(b). Using the graph, decide whether the integral of each of the functions $f(x)$, $g(x)$, $h(x)$ and $k(x)$ on the interval from $x = 1$ to ∞ converges, diverges, or whether it is impossible to tell.

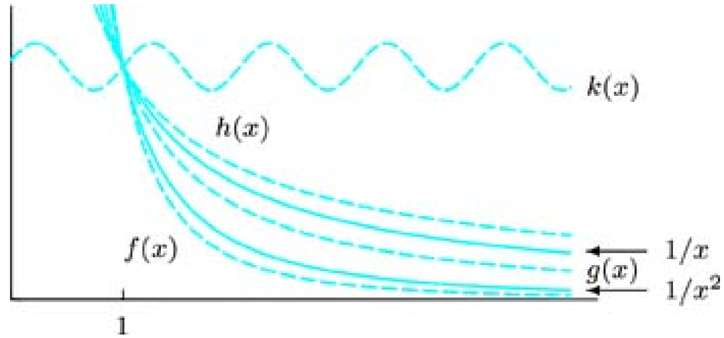

Figure 7.25:

29. Suppose $\int_a^{\infty} f(x)\,dx$ converges. What does Figure 7.26 suggest about the convergence of $\int_a^{\infty} g(x)\,dx$?

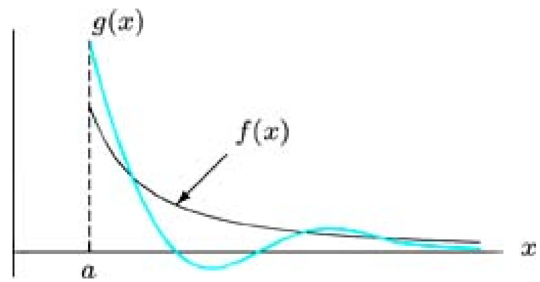

Figure 7.26:

For what values of *p* do the integrals in Problems 30–31 converge or diverge?

30. $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

31. $\int_1^{2} \frac{dx}{x(\ln x)^p}$

32. Find the value of *a* (to three decimal places) that makes

$$\int_{-\infty}^{\infty} ae^{-x^2/2}dx = 1 .$$

33.

(a). The function $g(x) = ae^{-(x-k)^2/2}$ is used by statisticians.To three decimal places, what value of *a* should be chosen to ensure that

$$\int_{-\infty}^{\infty} g(x)dx = 1 \text{ ?}$$

(b). Is your answer the same as or different from your answer to Problem 32? Why?

34.

(a). Find an upper bound for

$$\int_{3}^{\infty} e^{-x^2} dx .$$

[Hint: $e^{-x^2} \leq e^{-3x}$ for $x \geq 3$.]

(b). For any positive *n*, generalize the result of part (a) to find an upper bound for

$$\int_{n}^{\infty} e^{-x^2} dx$$

by noting that $nx \leq x^2$ for $x \geq n$.

35. In Planck's Radiation Law, we encounter the integral

$$\int_{1}^{\infty} \frac{dx}{x^5(e^{1/x} - 1)} .$$

(a). Explain why a graph of the tangent line to e^t at $t = 0$ tells us that for all t

$$1 + t \le e^t .$$

(b). Substituting $t = 1/x$, show that for all x

$$e^{1/x} - 1 > \frac{1}{x} .$$

(c). Use the comparison test to show that the original integral converges.