ASYMPTOTICS OF PARTITIONS

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Casey Warmbrand

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____________________ Date: 17 January 2007
Kenneth McLaughlin

____________________ Date: 17 January 2007
Nicholas Ercolani

____________________ Date: 17 January 2007
Hermann Flaschka

____________________ Date: 17 January 2007
Momar Dieng

____________________ Date: 17 January 2007

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Dissertation Director: Kenneth McLaughlin

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SIGNED: ______________________________

Casey Warmbrand

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This thesis has been approved on the date shown below:

______________________________ ____________________

Kenneth McLaughlin
Professor of Mathematics

Date

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ABSTRACT

Have you ever wondered what you would be likely to get if you randomly partitioned an object into infinitely many pieces? This paper will aim to answer this question. We will show, that when properly ordered, an infinite random partition will almost surely not look random at all. We view partitions as shapes of Young tableaux, for which there is a natural measure. Using this measure to answer our probabilistic question of randomness, we are able to find a unique shape as the size of our partition goes to infinity. The limiting curve we obtain, is a smooth function whose derivative can be interpreted as the likely hood of changing direction while on an infinite walk.

CHAPTER 1

Introduction and Preliminary Definitions

1.1 Introduction

This paper will consist of a detailed exposition of "Asymptotics of the Maximal and Typical Dimension of Irreducible Representations of the Symmetric Group," [4] a paper by A. Vershik and S. Kerov.

Irreducible representations of the symmetric group on n objects can be associated with partitions of n. The dimension of the irreducible representation associated to a partition, which we refer to as simply the dimension of the partition associated with it, can be extracted from the Plancherel measure of that partition.

Studying asymptotics, we allow the parameter n to tend toward infinity, and examine the behaviour of the Plancherel measure on the partitions of n. Specifically, we will obtain an upper bound for the maximal dimension of any partition and a lower bound for the dimension of almost every partition of n, with respect to the Plancherel measure.

The construction of the Plancherel measure comes from the Robinson-Schensted algorithm and resulting correspondence [2]. This correspondence leads us to a function which maps the symmetric group on n objects onto the set of partitions of n. The pushforward of the uniform measure on the symmetric group via this function is the resulting Plancherel measure on partitions. The Plancherel measure is given in terms of the dimension of the partition.

Another result, which can be found in "The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions" by B. Sagan [2], gives a way of calculating the dimension of a partition from its visual representation as a Ferrers diagram or Young tableau. It is called the hook length formula and is given here as equation (2.1).

The hook length formula gives the dimension of a partition in terms of a product of the hook lengths of each of the n boxes in the visual representation as a Ferrers diagram. So in taking a logarithm of the Plancherel measure we obtain a sum of logarithms of the individual hook lengths. It is from this sum that we are able to obtain a Riemann integral when taking a large n limit to study the asymptotics. This integral is referred to as the hook integral (2.6).

The Plancherel measure of a partition is asymptotically the same as the exponential of negative n times the hook integral of the partition. In trying to understand the asymptotics of the Plancherel measure, we wish to find the partitions of a large n which have the highest probability. In order to do this we must find the minimizer of the hook integral. The minimizer is given in equation (1.2) and in chapter 4 this function is shown to be the unique asymptotic minimizer of the hook integral.

The process of arriving at this minimizer involves a number of different interpretations of the partitions themselves as well as their visualizations. One view of a partition that we utilize is as a random walk in the plane, from a starting height determined by the size of the largest element of the partition to an ending height determined by the number elements (not necessarily distinct) in the partition. In the coordinates that make the partition appear as a random walk, the hook integral can be written in quadratic form (3.3).

Already knowing what the minimizer should look like, we next use a first variation to find the critical point of the hook integral. The variational argument turns our hook integral into something very similar to a weighted energy integral (3.8). Once the hook integral is in this form, it is immediately clear that the function Ω is a minimizer. We show that it is in fact the unique minimizer, in the same way that many weighted energy integrals are shown to have unique minimizers for their energy.

The minimizer can be viewed as a limit of a sequence of partitions of n as n tends toward infinity. In this sense we can think of it as an "infinite" partition. The value of the hook integral at the minimizer is 0, which means that the entire Plancherel measure is asymptotically supported there. Because of this, by bounding the hook integral of a partition, we are able to bound the Plancherel measure of that partition. This leads us to a bound on the dimension of the partition and its associated irreducible representation of the symmetric group.

We are able to give two different results. In the first we take n to be large, but fixed, and give an upper bound on the dimension of any partition of that size. The second result is given in terms of a limit as n tends toward infinity of the measure of a set of partitions whose dimensions are bounded below. The asymptotics of this second result lead us to the interpretation of this bound on "typical" partition.

1.2 Partitions and Young Tableaux

Let S_n denote the group of permutations on the set $\{1, 2, \ldots, n\}$.

Definition 1.2.1 *Given a positive integer n, we define a partition of n as a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$ and $\sum_i \lambda_i = n$. We write $\lambda \vdash n$, for λ a partition of n.*

We will adopt the convention that if $n = 0$, $\lambda = \emptyset$, is the empty partition. Note that we could have chosen to define a partition as an infinite, non-negative, non-increasing, sequence, $(\lambda_1, \lambda_2, \ldots)$ under the condition that only finitely many terms are non-zero. These definitions are equivalent for our purposes.

Definition 1.2.2 *Given two partitions $\alpha = (a_1, \ldots, a_l)$ and $\beta = (b_1, \ldots, b_m)$, we say β is a subpartition of α or that α covers β, if $m \leq l$, and $b_i \leq a_i$ for $1 \leq i \leq m$.*

Definition 1.2.3 *A Ferrers diagram associated to a partition, $\lambda \vdash n$ is an array of n dots arranged in left justified rows such that the i^{th} row has exactly λ_i dots.*

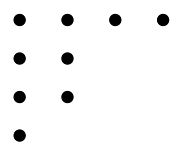

Figure 1.1: The Ferrers diagram above is an example for the partition $\lambda = (4, 2, 2, 1)$.

It would be equivalent to think of a Ferrers diagram consisting of boxes rather than dots. This will allow us to place numbers into the boxes and obtain a Young tableau. We will sometimes use the term *shape* when we are referring only to the configuration of dots or boxes. β is a subpartition of α, denoted $\beta \subseteq \alpha$ if, when superimposed, the shape of α covers the shape of β.

Definition 1.2.4 *A Young Tableau of shape λ is a Ferrers diagram where the dots are replaced by boxes and the numbers $1, 2, \ldots, n$ are placed into the boxes bijectively.*

4	1	8	7
6	5		
9	2		
3			

Figure 1.2: This is a Young tableau of shape $\lambda = (4, 2, 2, 1)$.

Given a Young tableau, Y we will denote the box in the ith row and jth column by $Y_{i,j}$ and the shape of Y by $\lambda(Y)$. We will use the term *size* to refer to the number n, that the shape λ, partitions.

Definition 1.2.5 *A box $Y_{i,j}$ will be called a corner if it lies at the end of a row and bottom of a column. That is to say $i = \lambda_k$ and $\lambda_k - \lambda_{k+1} > 0$ for some $k \in \{1, \ldots, l\}$.*

Earlier we mentioned that it would be equivalent to allow λ to be an infinite sequence, as long as the tail is all 0's. Another equivalent way to represent a partition, is by writing the terms of the sequence as $(\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots, \lambda_k^{p_k})$, where each λ_j is distinct, p_j is the number of times that λ_j appeared, and k is the number of distinct terms in the sequence λ. For example the partition (5,5,3,3,3,1,1) of 21, can be written as $(5^2, 3^3, 2^2)$. Another way to express these p_j's, is in terms of the indices of the rows which have corners.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we will find an increasing sequence of integers i_j, which are the indices of the rows in the original Young tableau which have corners. From these i_j's we will easily be able to find the number of times any term of λ is repeated.

Set $\lambda_0 = \infty, i_0 = 0$, and proceed by induction on j.

For $j = 1, \ldots, k$ set

$i_j =$ minimum index i, such that $i \geq i_{j-1}$ and $\lambda_{i_j} - \lambda_{i_j+1} > 0$.

Note that $k \leq l$, with equality if and only if all the terms of λ are distinct. Then $p_j = i_j - i_{j-1}$, is the number of times that λ_j is repeated in λ. Now that the partition λ can be written without repeats, we know:

$\forall k_1, k_2 \in \{i_{j-1}+1, i_{j-1}+2, \ldots, i_j - 1, i_j\}, \lambda_{k_1} = \lambda_{k_2}$.

This implies that $\lambda_1 = \lambda_{i_1}$ and $\lambda_{i_j+1} = \lambda_{i_{j+1}} \forall i_j, j = 1, \ldots k$. This representation of partitions will be useful to us in chapter 3, when we view the Young tableaux in a couple of different coordinate systems. Having the indices, i_j of the rows which have corners, will give us the coordinates of the corner boxes.

1.3 Standard Young Tableaux

Let YT^n_λ denote the set of all Young tableau of size n and shape λ. We may now pose the question, what is the cardinality of YT^n_λ? In order to create a Young tableau of shape $\lambda \vdash n$, we must choose λ_i of the numbers $\{1, \ldots, n\}$ to go into the i^{th} row of the tableau, and arrange them in some order within that row, for every row $1 \leq i \leq l$. For $0 \leq k \leq l$, let $s_k = \sum_{i=1}^k \lambda_i$, where $s_0 := 0$ and $s_l = n$. Given that the number of ways of choosing k objects from a set of m objects is $\binom{m}{k} = \frac{m!}{(m-k)!k!}$

and the number of ways of arranging k objects is $k!$, we have the following formula:

$$|YT^n_\lambda| = \prod_{m=1}^{l} \binom{n - s_{m-1}}{\lambda_m} \lambda_m! = n!.$$

The right hand side is obtained after proper cancellation. Note that this quantity is independent of the partition's shape, λ; it only depends on the size of the partition, n. It turns out to also be the number of permutations in S_n; and as expected, there is a simple bijection between these two sets. The bijection can be described, as follows: For any $\sigma \in S_n$, take a tableau of shape λ and place into each row $1 \le j \le l, \sigma(s_{j-1}+1), \sigma(s_{j-1}+2), \ldots, \sigma(s_j)$ in that order. Given a tableau we could easily represent σ by reading off the numbers in the tableau from left to right in the rows and top to bottom in the columns.

The question of the number of possible Young tableaux of a given shape λ, was not so interesting, as it did not depend on the associated partition, λ. We next consider the following subset of Young tableaux of size n and shape λ, whose cardinality will depend on the underlying partition:

Definition 1.3.1 *A Young tableau, Y is said to be standard if each row and each column is an increasing sequence: $Y_{i,j} < Y_{i,j+1}$ for $1 \le i \le \lambda_i$ and $1 \le j < l$ and $Y_{i,j} < Y_{i+1,j}$ for $1 \le i < \lambda_i$ and $1 \le j \le l$.*

1	4	7	8
2	5		
3	6		
9			

Figure 1.3: This is a standard Young tableau of shape $\lambda = (4, 2, 2, 1)$.

We will denote the set of standard Young tableaux of size n, SYT^n and the subset which has shape $\lambda \vdash n$, SYT^n_λ.

Definition 1.3.2 *The number of standard Young tableaux of size* n *and shape* $\lambda, |SYT_\lambda^n|$ *is called the* *dimension of* λ, *denoted* $dim(\lambda)$.

The number of standard Young tableaux of a given size is a combinatorial object. So you may be asking yourself why we call it the *dimension* of the partition. As we will see in the next section, the partitions of n are closely related to the permutations in S_n. Each partition of n is associated with an irreducible representation of S_n; and the dimension of this irreducible representation is the same as the number of standard Young tableaux of that shape. A good reference for the relationship between partitions of n and representations of the symmetric group on the set $\{1, \ldots, n\}$ is Bruce Sagan's text "The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions" [2].

1.4 The Robinson-Schensted Correspondence

The question of the number of *standard* Young tableau of shape λ is slightly more challenging than that of the number of Young tableaux. We will present two independent ways of viewing $dim(\lambda)$ in section (2). We first present an enumeration as part of a rooted tree. And then the *hook formula*, which calculates $dim(\lambda)$ directly from the partition λ.

The existence of a bijection between the permutations in S_n and pairs of standard Young tableaux of size n will allow us to relate the cardinalities of these two sets. This will result in an equation allowing us to construct a natural probability measure on the set of partitions. This bijection is a direct consequence of the following algorithm credited to Robinson and Schensted, and developed by Knuth:

Algorithm 1.4.1 (The Robinson-Schensted Algorithm) *Given a permutation* $\sigma \in S_n$, *written as* $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$, *we will construct a sequence of pairs of partial (may contain only some of the numbers* $1, \ldots, n$*) Young tableaux,* $(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), \ldots, (P_n, Q_n) = P(\sigma), Q(\sigma)) = RS(\sigma)$. *Each partial*

Young tableau will have distinct entries that are increasing across the rows and down the columns.

The algorithm will involve the action of inserting a number into a row of some P_k to obtain P_{k+1}, and then updating Q_k in a corresponding manner to obtain Q_{k+1}. We first describe the general procedure of inserting any number s into any row i of P_k:
If row i of P_k is empty, add a new box in position $(i, 1)$ and insert s into it to obtain P_{k+1}.
Otherwise, row i is not empty. We denote the i^{th} row of P_k by $(t_1, t_2, \ldots, t_m)$, where m is the length of the i^{th} row of P_k.
If $s > t_j$ $\forall j \in \{1, \ldots, m\}$, add a new box in position $(i, m + 1)$ and insert s into it to obtain P_k.
Otherwise, $\exists j \in \{1, \ldots, m\}$ such that $s < t_j$, take j minimal. In this case, the t_j gets bumped *by s, and we loop to insert the bumped number. Replace t_j with s in position (i, j) of P_k. Then insert t_j into the $(i + 1)^{th}$ row of P_k.*

After P_{k+1} has been created, we will need to get Q_{k+1} from Q_k. We do this by first adding one new box, so that $\lambda(Q_{k+1}) = \lambda(P_{k+1})$, and then placing the number $k + 1$ into it.

To run the algorithm, we initialize $P_0 = Q_0 = \emptyset$, the empty partition.
Loop over $k = 1, \ldots, n$:
Add $\sigma(k)$ to row 1 of P_{k-1} to get P_k. Add k to Q_{k-1} to get Q_k.

A few facts are immediately clear from the construction. We see that $\forall k \in \{0, \ldots, n\}, \lambda(P_k) = \lambda(Q_k) \vdash k$. That is both P_k and Q_k have the same shape, which is a partition of k. The shape of P_k will cover the shape of P_{k-1} for all $k \in \{1, \ldots, n\}$. We add the numbers $1, \ldots, n$ into the P's in the order $\sigma(1), \ldots, \sigma(n)$, therefore all of the P_k's will be standard if and only if $\sigma = id \in S_n$. And since we add the numbers $1, \ldots, n$ in order, into the Q's, Q_k will be a standard Young tableau $\forall k \in \{0, \ldots, n\}$. $Q(\sigma)$ is called the recording tableau and simply keeps track of the order that the boxes were added while creating $P(\sigma)$. This allows us to easily invert the algorithm to obtain a permutation in S_n from any pair of standard Young tableaux of the same shape. Since inversion is possible, the algorithm leads us to the following bijection:

Theorem 1.4.2 (The Robinson-Schensted Correspondence) *The following map is a bijection between permutations $\sigma \in S_n$ and pairs (P,Q) of standard Young tableaux of the same shape, λ:*

$$RS : \sigma \rightarrow (P(\sigma), Q(\sigma)).$$

Note that $P(\sigma)$ and $Q(\sigma)$ have the same shape, denoted $\lambda_{RS}(\sigma)$. A direct consequence of this bijection is that $\sum_{\lambda \vdash n} [dim(\lambda)]^2 = n!$.

1.5 The Plancherel Measure

Taking the uniform distribution on S_n, we can use the above bijection, RS to push it forward to a measure on the shapes of Young Tableau. Remember that shapes are simply partitions of n. Thus we obtain a measure on partitions, called the Plancherel measure. It is given by

$$\mu_n(\lambda) = \frac{[dim(\lambda)]^2}{n!}, \tag{1.1}$$

for $\lambda \vdash n$. Note that this is a probability measure as $\sum_{\lambda \vdash n} \mu_n(\lambda) = 1$.

We are now interested in an asymptotic analysis of this measure, for $n \rightarrow \infty$. We consider choosing a random permutation from S_n. The Robinson-Schensted algorithm gives us partition of n associated with the permutation. The partition can be thought of as the shape of the Young tableaux that the algorithm outputs.

Our problem asks whether there is a particular shape that emerges in the large n limit, and if so, what shape is it?

We will show that in the sense of the Plancherel measure, with probability 1, a unique shape emerges in the large n limit. However, the shape is no longer jagged like the boundary of a Young tableau; as $n \to \infty$, after an appropriate rescaling, the limiting curve is smooth. It's derivative will be of particular importance in some later applications. The curve we obtain, and it's derivative are presented below:

$$\Omega(X) = \begin{cases} \frac{2}{\pi}(X \arcsin(X) + \sqrt{1 - X^2}) & |X| \leq 1 \\ |X| & |X| > 1. \end{cases} \tag{1.2}$$

$$\Omega'(X) = \frac{2}{\pi} \arcsin(X), \tag{1.3}$$

in which $\Omega'(X)$ is naturally extended by ± 1 for $|X| \geq 1$.

CHAPTER 2

The Hook Formula and Large N Analysis

Before we can begin to analyze the Plancherel measure, we need some way of calculating $dim(\lambda)$. In this section we will first present a combinatorial enumeration of a family of standard Young tableau related to the partition λ. Next we will present the hook formula, which will allow us to proceed with the large n analysis. We conclude the section by constructing the *hook integral*, which will be a key object in the computation of $\lim_{n\to\infty} \mu_n(\lambda)$.

2.1 A Partition Tree

Given a partition $\lambda \vdash n$, we construct a rooted tree, $T(\lambda) = \{V, r, p\}$. The nodes of the tree, $V \cup \{r\}$, are distinct standard Young tableaux, and p is a function from V to $V \cup \{r\}$ which assigns to each non-root node $v \in V$, a unique parent node in the tree. The root node r, is the empty tableau (the tableau corresponding to the empty partition, $\emptyset$). The set of non-root nodes is defined as follows:

$V = \{SYT^m_\alpha : 1 \leq m \leq n, \alpha \vdash m, \text{ and } \alpha \subseteq \lambda\}.$

We define the parent of a node $v \in V$ of size m, as:

$p(v) =$ The tableau obtained by removing the box containing m from v.

Note that $\lambda(p(v)) \subseteq \lambda(v)$ and the size of v equals the size of $p(v) + 1, \forall v \in V$. The *depth* of a node, $v \in V$, is the minimal distance from the root r to v. The depth of v is exactly the size of v. The number of nodes at a depth of n from the root, is exactly $dim(\lambda)$.

This is a nice combinatorial construction, which is helpful in visualizing how one would count the number of standard Young tableaux of shape $\lambda \vdash n$. However, this does not provide us with a closed formula for $dim(\lambda)$, to be analyzed in the large n limit. Fortunately, such a formula exists; it is known as the *Hook Length Formula.*

2.2 Hook Length

Definition 2.2.1 *Given a partition λ, the hook length, $h_{i,j}$ of the position $Y_{i,j}$ of any corresponding Young Tableau, is the number of boxes to the right of $Y_{i,j}$ (the arm) , plus the number of boxes below $Y_{i,j}$ (the leg), plus one for the box $Y_{i,j}$ itself.*

7	5	2	1
4	2		
3	1		
1			

Figure 2.1: For each box in the shape $\lambda = (4, 2, 2, 1)$, we have placed the hook length, $h_{i,j}$ in box (i, j).

We point out the fact that a box has hook length 1 if and only if it is located at the end of a row and the bottom of a column, i.e. the box is a corner.

Theorem 2.2.2 (The Hook Length Formula) *The number of standard Young tableaux of a particular shape $\lambda \vdash n$ is given by:*

$$dim(\lambda) = \frac{n!}{\prod_{i,j} h_{i,j}} \text{ for } 1 \leq i \leq l \text{ and } 1 \leq j \leq \lambda_i. \tag{2.1}$$

A proof of this theorem can be found in [2]. Using this formula, we may rewrite the Plancherel measure as follows:

$$\mu_n(\lambda) = \frac{dim^2(\lambda)}{n!} = \frac{n!}{\prod_{i,j} h_{i,j}^2}. \tag{2.2}$$

This allows us to begin our large n analysis. In thinking about asymptotics as $n \to \infty$, we will consider a sequence of partitions $\Lambda_1, \Lambda_2, \ldots$ This sequence will have restrictions necessary for convergence of the boundary shape. Next, we approximate the $n!$ in (2.2) with Stirling's approximation:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}\pm\dots\right) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right),$$

where a function f is $O(g(n))$ if there exist positive constants N and C, such that $\forall n > N, |f(n)| \leq Cg(n)$. Since the error term from Stirling's approximation is $O\left(\frac{1}{n}\right)$ and depends only on n, it must go to 0 as $n \to \infty$. Applying Stirling's approximation to (2.2) and then taking a logarithm, gives us:

$$\ln \mu_n(\lambda) = \ln\left[\sqrt{2\pi n}\left(1+O\left(\frac{1}{n}\right)\right)\right] - n + n\ln n - 2\sum_{i,j}\ln h_{i,j}.$$

Since the sum over i and j has precisely n terms (one for each box in the shape λ), the last two terms can be combined into $2\sum_{i,j}\ln\left(\frac{\sqrt{n}}{h_{i,j}}\right)$. Finally, we divide by n, and negate, leaving us with:

$$-\frac{\ln \mu_n(\lambda)}{n} = 1 + \frac{2}{n}\sum_{i,j}\ln\left(\frac{h_{i,j}}{\sqrt{n}}\right) - \epsilon_n := J_\lambda - \epsilon_n, \tag{2.3}$$

where the error from Stirling's approximation, $\epsilon_n = \ln n! - n\ln\frac{n}{e} - \ln\sqrt{2\pi n}$, depends only on n and is $O\left(\frac{\ln n}{n}\right)$, thus $\epsilon_n \to 0$ as $n \to \infty$.

2.3 The Hook Integral

We are now concerned with asymptotics. As we will show in this section, in the limit as $n \to \infty$, the term J_λ will turn into an integral. In order to see this convergence, we first consider the boundary of the shape of a Young tableau. For convenience, we place the tableau in the the first quadrant of the plane, so that the top of the first row coincides with the $y-$axis and the left side of the first column coincides with the $x-$axis, so that the upper left corner of the tableau is at the origin. Note that this rotates the lower right corner of the box at position (i, j) in the original Young tableau to be in the upper right corner at coordinates (i, j) in the plane. The boundary of the tableau encloses a region in the first quadrant, of area n. Let s_λ denote this region in the plane. Since there are n boxes inside s_λ, each with area 1, the shape of λ places a grid on s_λ.

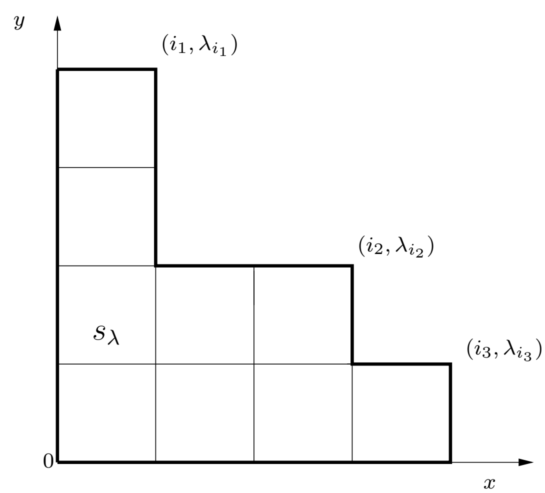

Figure 2.2: The partition $\lambda = (4, 2, 2, 1)$ drawn in the $(x, y)-$coordinates.

As $n \to \infty$, the region s_λ becomes unbounded. Thus, we introduce a normalization so that the area of $s_\lambda = 1, \forall n$. To do this we simply contract the tableau by $\sqrt{n}$ along both axes. Now any box (i, j) will have a corner at the point $(\frac{i}{\sqrt{n}}, \frac{j}{\sqrt{n}}) = \frac{1}{\sqrt{n}}(i, j)$, have sides of length $\frac{1}{\sqrt{n}}$, and have area of $\frac{1}{n}$. In rescaling the tableau, now take the rescaled hook of box (i, j) to be $\frac{h_{i,j}}{\sqrt{n}}$.

To aid in the analysis we will define a function to model the boundary of the tableau. First consider only the jagged edge of the boundary. This now looks like stairs from the point on the $y-$axis, $\frac{1}{\sqrt{n}}(0, \lambda_1)$ to the point on the $x-$axis, $\frac{1}{\sqrt{n}}(l, 0)$ (recall that l is the number of rows of the tableau). We will extend this curve to include the $y-$axis between $\frac{1}{\sqrt{n}}(0, \lambda_1)$ and $\frac{1}{\sqrt{n}}(0, n)$ and the $x-$axis between $\frac{1}{\sqrt{n}}(l, 0)$ and $\frac{1}{\sqrt{n}}(n, 0)$. We will denote this infinite curve by γ, and from it define a function. The curve γ consists of line segments of length $\frac{1}{\sqrt{n}}$, which are either horizontal or vertical. On the left ($x = 0$) they are *all* vertical and on the right (x sufficiently large) they are *all* horizontal. In order to define a function, we must specify which value to take where the curve is vertical. We let $F_\lambda(x) = \inf \{\text{height of } \gamma \text{ at } x\}\ \forall x \in [0, \infty)$. We further define an inverse, $F_\lambda^{-1}(x) = \inf \{x : F_\lambda(x) \leq y\}$.

We define the hook of the point (x, y) in s_λ as:

$$h_{F_\lambda}(x, y) := (F_\lambda(x) - y) + (F_\lambda^{-1}(y) - x).$$

The first term coincides with the length of the *arm* and the second with the length of the *leg* in definition 2.2.1. We also have:

$$s_\lambda := \{(x, y) : 0 \leq x \leq \infty, 0 \leq y \leq F_\lambda(x)\}.$$

Given any finite bounded non-increasing function $y = F(x)$, we define $F^{-1}(y) = \inf \{x : F(x) \leq y\}, h_F = F(x) - y + F^{-1}(y) - x$, and $s_F = \{(x, y) : 0 \leq x < \infty, 0 \leq y < F(x)\}$. We will now consider a sequence of functions $F_{\Lambda_n} \to F$, modeling the boundaries of a corresponding sequence of partitions $\Lambda_1, \Lambda_2, \ldots$ This sequence of partitions is not arbitrary, it must be chosen in such a way that the corresponding sequence $\{F_{\Lambda_i}\}$ converges.

In the rescaled tableau, the grid spacing is now $\frac{1}{\sqrt{n}}$, so we let $\Delta x = \Delta y = \frac{1}{\sqrt{n}}$. We let $(x_i, y_j) := \frac{1}{\sqrt{n}}(i, j)$. The rescaled hook of box (i, j) in a tableau of shape Λ_n is now equal to the hook at the point (x_i, y_j) for the function F_{Λ_n}:

$$\frac{h_{i,j}}{\sqrt{n}} = h_{F_{\Lambda_n}}(x_i, y_j) \tag{2.4}$$

Substituting into equation (2.3), we obtain:

$$J_{\Lambda_n} = 1 + 2 \sum_{i,j} \ln \big(h_{F_{\Lambda_n}}(x_i, y_j)\big) \Delta x \Delta y, \tag{2.5}$$

where the sum over i, j refers to all of the boxes (i, j) of a tableau with shape Λ_n.

The term J_{Λ_n} now looks like a Riemann sum. Since F_{Λ_n} converges to a function, in the limit $n \to \infty$ we obtain the following integral:

$$\theta(F) := 1 + 2 \iint_{s_F} \ln h_F(x, y) dx dy, \tag{2.6}$$

where dx and dy are Lebesgue measure on the real line. We call this the *hook integral.*

The following formula involving the Plancherel measure, show this convergence:

$$-\lim_{n\to\infty} \frac{\ln \mu_n(\Lambda_n)}{n} = \lim_{n\to\infty} \{J_{\Lambda_n} - \epsilon_n\} = \theta(F) \tag{2.7}$$

A complete proof showing the decay of the error terms may be found in [4].

With this result, our large n analysis of the Plancherel measure has been reduced to an analysis of the hook integral. We can see from the second equality in (2.7), $\lim_{n\to\infty} \mu_n(\Lambda_n)$ will be 0, unless $J_{\Lambda_n} \to 0$ as $n \to \infty$. We will see in chapter 3, the function Ω from (1.2) is the unique minimizer of θ, i.e. $\theta(\Omega) = 0$. We already see that because μ is a probability measure, $\mu(\lambda) \leq 1, \forall\lambda$ which implies that $\theta(F) \geq 0$, by the first equality in (2.7). So, the hook integral is always non-negative.

CHAPTER 3

Analysis of the Hook Integral

This chapter will cover the details involved in transforming the hook integral into quadratic form. In the first section, we will once again change our view of a partition. We will now think of a partition as a random walk in the upper half-plane. The change of coordinates that accomplishes this, will transform our function modeling the boundary as well as the hook integral. The second section will introduce a final change of coordinates, which do not seem to have as nice visual interpretation as all the preceding coordinate systems. In the end, the hook integral will be in the form of an energy problem. At that point, the only thing left in order to find our desired function Ω, will be to minimize this energy problem. The minimization is left to chapter 4.

3.1 Partitions Appearing as Random Walks

We first need to apply a simple change of coordinates to our current system (x, y) in which $y = F_\lambda(x)$ is defined. The new coordinate system is $\phi(x, y) = (X, Y) = (\frac{x-y}{2}, \frac{x+y}{2}) = \frac{1}{2}(x-y, x+y)$. Note that this implies $\phi^{-1}(X, Y) = (x, y) = (X+Y, Y-X)$. In the new coordinate system the old positive $x-$ and $y-$axes correspond to the lines $Y = X$ and $Y = -X$, respectively. This is a counter-clockwise rotation of $45°$and a rescaling that takes a point distance d from the origin to a point distance $\frac{d}{\sqrt{2}}$ from the origin.

Consider what happens to the boundary of a Young tableau of shape $\lambda \vdash n$. Previously, the boundary was the curve we called γ; a series of length $\frac{1}{\sqrt{n}}$ lines moving either horizontally to the right or vertically down from the point $\frac{1}{\sqrt{n}}(0, \lambda_1)$

on the $y-$axis to the point $\frac{1}{\sqrt{n}}(l,0)$ on the $x-$axis. The curve γ was equivalent to the function F_λ. In the new $(X,Y)-$coordinates, the boundary will be a series of lines with derivative ± 1, starting from $\frac{1}{2\sqrt{n}}(-\lambda_1,\lambda_1)$ and going to the point $\frac{1}{2\sqrt{n}}(l,l)$. We can naturally extend the boundary to be over the entire real line, by letting it coincide with the image of the $y-$axis on the left and the image of the $x-$axis on the right. We shall denote this new boundary $Y = L_\lambda(X)$. The properties that F_λ satisfies translate into the following properties that L_λ must satisfy:

$$\begin{aligned} L'_\lambda(X) &= \pm 1, \text{ for almost every X} \\ L_\lambda(X) &\geq |X|, \text{ with equality for all sufficiently large } |X|. \end{aligned} \tag{3.1}$$

The function L_λ is simply the image of the infinite curve γ under the above transformation. Define $r_\lambda := \{(X,Y) : |X| \leq Y \leq L_\lambda(X)\}$. The region r_λ is the image of the region s_λ.

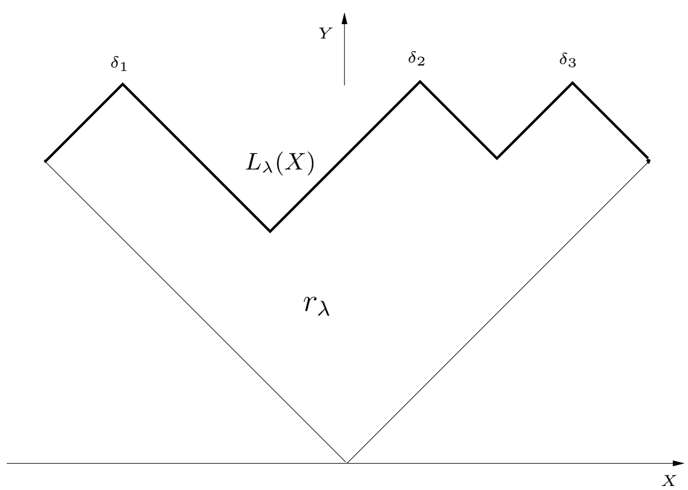

Figure 3.1: The partition $\lambda = (4,2,2,1)$ drawn in the $(X,Y)-$coordinates. The boundary indicated by the dark line, is the curve $L_\lambda(X)$. The downturns (corners) are indicated by the δ_i's.

As we will soon see, this coordinate system allows us to view this new boundary as a random walk from a height on the line $Y=-X$ to a height on the line $Y=X$.

Recall the representation of a partition that was presented after definition 1.2.5, in which all the terms were distinct. We wrote the partition as $\lambda = (\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots, \lambda_k^{p_k})$ where each λ_j is distinct, p_j is the number of times λ_j was repeated in λ, and k is the number of distinct terms in λ. Also recall that in finding the p_j's, we found the indexes, i_j of the rows which have corners.

Viewing the tableau in the $(x, y)-$coordinates from section 2.3, it looks like a flight of stairs going down, each corner corresponding to a step. The corners occur at the coordinates $\frac{1}{\sqrt{n}}(i_j, \lambda_{i_j})$. Each down step is of depth $\frac{1}{\sqrt{n}}(\lambda_{i_j} - \lambda_{i_{j+1}})$, and goes from $\frac{1}{\sqrt{n}}(i_j, \lambda_{i_j})$ to $\frac{1}{\sqrt{n}}(i_j, \lambda_{i_{j+1}})$.

After applying the transformation ϕ, the boundary of a tableau can now be thought of as a random walk. The walk can be thought of as starting at $\frac{1}{2\sqrt{n}}(-n, n)$ on the line $Y = -X$ and moving down one unit at each step until it reaches the point $\frac{1}{2\sqrt{n}}(-\lambda_1, \lambda_1)$, where it first turns upward. This point is the image of $\frac{1}{\sqrt{n}}(0, \lambda_1)$, where the tableau began in the old $(x, y)-$coordinates. The walk continues upward until it reaches the image of the first corner of the original Young tableau, where it turns downward for a number of steps proportionate to the height of the step at that corner in the $(x, y)-$coordinates. The walk then turns upward again until the image of the second corner is reached, where it turns down again for some number of steps proportionate to that corners height, and then turns back up. This pattern continues until we reach the line $Y = X$, and then potentially travels upward to the point $\frac{1}{2\sqrt{n}}(n, n)$. The walk coincides with the function L_λ. We will now find the coordinates of all the turning points throughout the walk.

Since the corners occurred at $\frac{1}{\sqrt{n}}(i_j, \lambda_{i_j})$, their images now occur at the points $\frac{1}{2\sqrt{n}}(i_j - \lambda_{i_j}, i_j + \lambda_{i_j})$. These are the places where the walk changes from up to down. We shall refer to these points as *down turning points*, denoted $\delta_j = \frac{1}{2\sqrt{n}}(i_j - \lambda_{i_j}, i_j + \lambda_{i_j})$. Since the height of the step at the corner located at $\frac{1}{\sqrt{n}}(i_j, \lambda_{i_j})$ was previously $\frac{1}{\sqrt{n}}(\lambda_{i_j} - \lambda_{i_{j+1}})$, we know that after turning downward at the point $\frac{1}{2\sqrt{n}}(i_j - \lambda_{i_j}, i_j + \lambda_{i_j})$, we continue down for $\frac{1}{2\sqrt{n}}(\lambda_{i_j} - \lambda_{i_{j+1}})$ many steps of the walk, at a slope of

-1. This takes us to the point $\frac{1}{2\sqrt{n}}(i_j - \lambda_{i_j} - 2\lambda_{i_{j+1}}, i_j - \lambda_{i_j} + 2\lambda_{ij+1})$. These are the places where the walk changes from down to up. These will be referred to as *up turning points*, denoted $u_j = \frac{1}{2\sqrt{n}}(i_j - \lambda_{i_j} - 2\lambda_{i_{j+1}}, i_j - \lambda_{i_j} + 2\lambda_{ij+1})$.

Just as we saw that all the information of λ could be expressed with the coordinates of the corners, we can also express λ using only the coordinates of the downs, δ_j (equivalently, only the ups, u_j). The set of downs (or the ups) will be of interest when trying to answer questions like: "What is the probability of having a specific sequence of ups and downs at each step in the walk?"

Now that we have described the boundary of a Young tableau as the function L_λ and seen how this models a random walk in the plane, we will consider one final change of coordinates. These coordinates do not appear to have as nice a geometric interpretation, but they greatly simplify the term inside the logarithm in equation (2.6), as well as the region of integration. We will show that this obscure change of variables is non-singular as well as orientation preserving. Then, after a small calculation, the hook integral from equation (2.6) will be in quadratic form.

3.2 Quadratic Form of the Hook Integral

The purpose of this section will be to introduce the final coordinates, (s, t) that we will use to write the hook integral. In these variables the hook length, $h_F(x, y)$ will look simply like $2(s - t)$. For now we will consider functions $y = F(x)$ which are strictly decreasing.

Define the coordinate change, ψ_F,

$$\psi_F(x, y) = (s, t) := \frac{1}{2}\left(F^{-1}(y) - y, x - F(x)\right).$$

It is immediately clear that,

$$2(s - t) = F(x) + F^{-1}(y) - x - y = h_F(x, y).$$

We next consider the Jacobian of this transformation.

$$\begin{aligned}\frac{\partial s}{\partial y} &= \frac{1}{2}\left(\frac{1}{F'(F^{-1}(y))} - 1\right) \\ \frac{\partial t}{\partial x} &= \frac{1}{2}\left(1 - F'(x)\right). \end{aligned} \tag{3.2}$$

The other two terms in the Jacobian are zero, and thus the Jacobian is the negative product of these two terms. To see that the transformation is non-singular, we must show that both of these terms are non-zero. Since the function $y = F(x)$ is strictly decreasing, its derivative $F'(x)$ must be negative everywhere. This means that $\frac{\partial s}{\partial y} < 0$ and $\frac{\partial t}{\partial x} > 0$ everywhere, and so the Jacobian is positive everywhere. This implies that this transformation is orientation preserving and does not compress any regions of space.

In the above discussion, we had assumed that $y = F(x)$ was strictly decreasing, whereas the F_λ from the previous chapter was either horizontal or vertical everywhere. We now consider any non-increasing function $y = F(x)$ and choose a sequence, F_n of strictly decreasing functions, such that $F_n \to F$. Define $\psi_F := \lim_{n\to\infty} \psi_{F_n}$. Since for each n, the transformation ψ_{F_n} is non-singular and orientation preserving, the limit will be as well. We now consider any finite bounded non-increasing function, $y = F(x)$.

For any x and for any y,

$$\begin{aligned}\phi(x, F(x)) &= \frac{1}{2}\left(x - F(x), x + F(x)\right) \\ \phi(F^{-1}(y), y) &= \frac{1}{2}\left(F^{-1}(y) - y, F^{-1}(y) + y\right),\end{aligned}$$

are the images of two points which lie on the function F. Therefore they lie on the function L_λ:

$$\begin{aligned}\frac{1}{2}\left(x-F(x),x+F(x)\right) &= \left(t,\frac{x+F(x)}{2}\right)=(t,L_\lambda(t))\\ \frac{1}{2}\left(F^{-1}(y)-y,F^{-1}(y)+y\right) &= \left(s,\frac{F^{-1}(y)+y}{2}\right)=(s,L_\lambda(s)).\end{aligned}$$

This gives us the following two equations for L_λ in the $(s,t)-$coordinates,

$$\begin{aligned}L_\lambda(t) &= \frac{x+F(x)}{2}\\ L_\lambda(s) &= \frac{F^{-1}(y)+y}{2}.\end{aligned}$$

And it is clear that,

$$\begin{aligned}L_\lambda(t)+t = & \tfrac{x+F(x)}{2}+\tfrac{x-F(x)}{2} &= x\\ L_\lambda(s)-s = & \tfrac{F^{-1}(y)+y}{2}-\tfrac{F^{-1}(y)-y}{2} &= y.\end{aligned}$$

Differentiating these two equations,

$$\begin{aligned}\frac{\partial x}{\partial t} &= L'_\lambda(t)+1\\ \frac{\partial y}{\partial s} &= L'_\lambda(s)-1.\end{aligned}$$

The Jacobian is the negative product of these, $(1-L'_\lambda(s))(1+L'_\lambda(t))$.

Substituting $2(s-t)$ for $h_F(x,y)$ and changing the variables from equation (2.6) we get the following alternative form of the hook integral:

$$\theta(L)=1+2\iint_{t<s}[\ln 2(s-t)](1-L'(s))(1+L'(t))\ dsdt. \tag{3.3}$$

The domain of integration above is $\{t<s\}$, because this is where the hook length, $2(s-t)$ is positive.

Let $Y = L(X)$ be any function which satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad L(X) \geq |X|, \text{ with equality for all sufficiently large } |X| \\ &\text{(ii)} \qquad\qquad -1 \leq L'(X) \leq 1, \text{ for almost every } X. \end{aligned} \tag{3.4}$$

We consider a sequence of functions L_{Λ_n} such that $L_{\Lambda_n} \rightarrow L$. In the $(s,t)-$coordinates this sequence models the boundaries of a corresponding sequence of partitions, $\Lambda_1, \Lambda_2, \ldots$. If F_{Λ_n} is another sequence that models the boundaries of this same sequence of partitions in the $(x,y)-$coordinates and $F_{\Lambda_n} \rightarrow F$, then the following is true:

$$\theta(F) = \lim_{n\to\infty} \{J_{\Lambda_n} - \epsilon_n\} = \lim_{n\to\infty} \theta(L_{\Lambda_n}) = \theta(L). \tag{3.5}$$

We note that the term $\theta(L_{\Lambda_n})$ is *not* a hook integral. This is a discrete term based on a partition of a finite n. It is a double sum in the $(X,Y)-$coordinates plus an appropriate error term from Stirling's approximation, much the same as J_{Λ_n} is a double sum in the $(x,y)-$coordinates. For n finite and $\lambda \vdash n$ we have,

$$\theta(L_\lambda) = -\frac{1}{n} \ln \mu_n(\lambda) = 1 + \frac{2}{n} \sum_{i,j} \ln \frac{h_{i,j}}{\sqrt{n}} - \epsilon_n.$$

We say that the formula in equation (3.3), represents the hook integral, θ in quadratic form. Note that because L satisfies property (ii) above, the integrand in (3.3) is 0 outside of some compact set, $S = [a,b] \times [a,b] = [a,b]^2$.

3.3 First Variation, Obtaining an Energy Problem

Recall the function from equation (1.2), which we claimed to be the minimizer of the hook integral:

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x \arcsin(x) + \sqrt{1-x^2}) & |x| \leq 1 \\ |x| & |x| > 1. \end{cases} \tag{3.6}$$

Given a function L which satisfies (3.4), we define a function f to measure the deviation from average as follows:

$$f(x) := L(x) - \Omega(x). \tag{3.7}$$

Because L satisfies the condition that $L(x) = |x|$ for all sufficiently large $|x|$, f has compact support. The remainder of this section describes the techniques used to write $\theta(L)$ in terms of f. The formula we are aiming for is:

$$\theta(L) = -\iint_S \ln 2|s-t| f'(s)f'(t)dsdt + 4\int_{|s|>1} f(s)\cosh^{-1}|s|ds, \tag{3.8}$$

where $S = [a,b]^2$ is a square which contains the support of f, and $\cosh^{-1}$ is the inverse of the hyperbolic cosine function.

To make the calculation simpler, we define the following functions for $n \geq 1$, recursively:

$$\Phi_0(x) := -\ln 2|x|, \quad \Phi_n(x) := \int_0^x \Phi_{n-1}(y)dy.$$

We note that Φ_0 is even, because of the absolute value in the definition. Using this we see that:

$$\Phi_1(-x) = \int_0^{-x} \Phi_0(y)dy = -\int_0^x \Phi(-y)dy = -\int_0^x \Phi_0(y)dy = -\Phi_1(x),$$

and so, Φ_1 is odd.

We will make use of the following property of these Φ_i's:

$$\begin{aligned}\int_t^b \Phi_0(s-t)ds - \int_a^t \Phi_0(s-t)ds &= \int_0^{b-t} \Phi_0(y)dy + \int_0^{a-t} \Phi_0(y)dy \\ &= \Phi_1(b-t) + \Phi_1(a-t).\end{aligned}$$

We can re-write the hook integral from (3.3) as follows:

$$\theta(L) = 1 - 2\iint_{s<t} \Phi_0(s-t)\left[1 - L'(s)L'(t)\right] dsdt - 2\iint_{s<t} \Phi_0(s-t)\left[L'(t) - L'(s)\right] dsdt.$$

We start by examining the first double integral above. The integrand is symmetric in s and t, and has compact support. Let $S = [a, b]^2$ be the compact region of integration:

$$\begin{aligned} & -\iint_{s<t} \Phi_0(s-t)\left[1 - L'(s)L'(t)\right] dsdt - \iint_{s<t} \Phi_0(s-t)\left[1 - L'(s)L'(t)\right] dsdt \\ = & -\iint_{s<t} \Phi_0(s-t)\left[1 - L'(s)L'(t)\right] dsdt - \iint_{t<s} \Phi_0(t-s)\left[1 - L'(t)L'(s)\right] dtds \\ = & -\iint_{\mathbb{R}^2} \Phi_0(s-t)\left[1 - L'(s)L'(t)\right] dsdt \\ = & -\iint_{S} \Phi_0(s-t)\left[1 - L'(s)L'(t)\right] dsdt. \end{aligned}$$

Consider the second double integral in $\theta(L)$ above:

$$\begin{aligned}
& -2\left[\iint_{t<s}\Phi_0(s-t)L'(t)dsdt-\iint_{t<s}\Phi_0(s-t)L'(s)dsdt\right] \\
= & -2\left[\iint_{t<s}\Phi_0(s-t)L'(t)dsdt-\iint_{s<t}\Phi_0(t-s)L'(t)dtds\right] \\
= & -2\left[\int_a^b L'(t)\int_t^b \Phi_0(s-t)dsdt-\int_a^b L'(t)\int_a^t \Phi_0(s-t)dsdt\right] \\
= & -2\int_a^b L'(t)\left[\int_t^b \Phi_0(s-t)ds-\int_a^t \Phi_0(s-t)ds\right]dt \\
= & -2\int_a^b L'(t)\left[\Phi_1(b-t)+\Phi_1(a-t)\right]dt
\end{aligned}$$

We now express the hook integral as:

$$\theta(L)=1-\iint_S \Phi_0(s-t)[1-L'(s)L'(t)]dsdt-2\int_a^b L'(t)[\Phi_1(b-t)+\Phi_1(a-t)]dt.$$

The first term of the first double integral is equal to $2\Phi_2(b-a)$:

$$\begin{aligned}
\int_a^b\left[\int_a^b \Phi_0(s-t)ds\right]dt &= \int_a^b\left[\int_0^{b-t}\Phi_0(y)dy+\int_{a-t}^0 \Phi_0(y)dy\right]dt \\
&= \int_a^b\left[\Phi_1(b-t)-\Phi_1(a-t)\right]dt \\
&= -\int_{b-a}^0 \Phi(y)dy+\int_0^{a-b}\Phi_1(y)dy \\
&= \Phi_2(b-a)+\Phi_2(a-b) \\
&= 2\Phi_2(b-a).
\end{aligned}$$

The last equality being due to the fact that Φ_2 is even (shown below, using that Φ_1 is odd):

$$\Phi_2(-x)=\int_0^{-x}\Phi_1(y)dy=-\int_0^x \Phi(-z)dz=\int_0^x \Phi(z)dz=\Phi_2(x).$$

We introduce another piece of notation:

$$I(s) := \int_a^b \Phi_0(s-t)\Omega'(t)dt,$$

The following two identities involving $I(s)$ will be used to complete the calculation. Their proofs are independent of the rest of the calculation and will be left to the appendix.

$$I(s) = \Phi_1(a-s) + \Phi_1(b-s) - 2H(s), \tag{3.9}$$

where $H(x) := x\cosh^{-1}|x| \mp \sqrt{x^2-1}$ for $\pm x \geq 1$ and $H(x) = 0$ for $|x| < 1$.

$$\int_a^b I(s)\Omega'(s)ds = 1 - 2\Phi_2(b-a) - 4\int_a^b H(s)\Omega'(s)ds. \tag{3.10}$$

Let $Q := \iint_S \Phi_0(s-t)L'(s)L'(t)dsdt$ and $R := \iint_S \Phi_0(s-t)\Omega'(s)\Omega'(t)dsdt$. Applying the identities above to our hook integral:

$$\begin{aligned}
\theta(L) &= 1 - \Phi_2(b-a) + Q - 2\int_a^b L'(s)[\Phi_1(b-s) + \Phi_1(a-s)]ds. \\
&= \int_a^b I(s)\Omega'(s)ds + 4\int_a^b H(s)\Omega'(s)ds - 2\int_a^b L'(s)[I(s) + 2H(s)]ds + Q \\
&= \int_a^b I(s)\Omega'(s)ds - 2\int_a^b I(s)L'(s)ds + Q + 4\int_a^b H(s)[\Omega'(s) - L'(s)]ds \\
&= \int_a^b I(s)\Omega'(s)ds - 2\int_a^b I(s)L'(s)ds + Q - 4\int_a^b H(s)f'(s)ds.
\end{aligned}$$

Applying the definition of $I(s)$ to the first two terms above:

$$\begin{aligned}
&\int_a^b I(s)\Omega'(s)ds - 2\int_a^b I(s)L'(s)ds \\
= &\iint_S \Phi_0(s-t)\Omega'(t)\Omega'(s)dtds - 2\iint_S \Phi_0(s-t)\Omega'(t)L'(s)dtds \\
= &\ R - 2\iint_S \Phi_0(s-t)L'(s)\Omega'(t)dtds.
\end{aligned}$$

Working from the other direction, the double integral in (3.8) can be written as:

$$\begin{aligned}
& -\iint_S \ln 2|s-t|f'(s)f'(t)dsdt \\
= \;& \iint_S \Phi_0(s-t)[\Omega'(s)\Omega'(t) - L'(s)\Omega'(t) - L'(t)\Omega'(s) - L'(s)L'(t)]dsdt \\
= \;& R - \iint_S \Phi_0(s-t)L'(s)\Omega'(t)dsdt - \iint_S \Phi_0(s-t)L'(t)\Omega'(s)dsdt + Q \\
= \;& R - 2\iint_S \Phi_0(s-t)L'(s)\Omega'(t)dsdt + Q
\end{aligned}$$

Putting this all together we have:

$$\theta(L) = -\iint_S \ln 2|s-t|f'(s)f'(t)dsdt - 4\int_a^b H(s)f'(s)ds. \tag{3.11}$$

Integrating by parts and using the fact that f is zero at both a and b, and H is zero for $|s| < 1$:

$$\begin{aligned}
-4\int_a^b H(s)f'(s)ds &= -4H(s)f(s)\Big|_a^b + 4\int_a^b f(s)H'(s)ds \\
&= 4\int_{|s|>1} f(s)H'(s)ds,
\end{aligned}$$

Writing $H(s)$ in logarithmic form using $\cosh^{-1}(x) = \ln\left[x \pm \sqrt{x^2-1}\right]$ we see that:

$$\begin{aligned}
H'(x) &= \cosh^{-1}|x| + x\frac{1 \pm \frac{x}{\sqrt{x^2-1}}}{x \pm \sqrt{x^2-1}} \mp \frac{x}{\sqrt{x^2-1}} \\
&= \cosh^{-1}|x| + x(1 \pm \frac{x}{\sqrt{x^2-1}})(x \mp \sqrt{x^2-1}) \mp \frac{x}{\sqrt{x^2-1}} \\
&= \cosh^{-1}|x| + x(\mp\sqrt{x^2-1} \pm \frac{x^2}{\sqrt{x^2-1}}) \mp \frac{x}{\sqrt{x^2-1}} \\
&= \cosh^{-1}|x| + x(\frac{\pm 1}{\sqrt{x^2-1}}) \mp \frac{x}{\sqrt{x^2-1}} \\
&= \cosh^{-1}|x|.
\end{aligned}$$

Since $H'(s) = \cosh^{-1}|s|$, we have arrived at equation (3.8):

$$\theta(L) = -\iint_S \ln 2|s-t| f'(s) f'(t) ds dt + 4 \int_{|s|>1} f(s) \cosh^{-1}|s| ds.$$

It is immediately clear that $\theta(\Omega) = 0$ because when $L = \Omega, f = 0$. In the next chapter we will see that Ω is the unique minimizer of the hook integral, θ. We will also obtain an upper bound for the dimension of any partition as well as a lower bound for the dimension of what are called the *essential partitions*.

CHAPTER 4

Uniqueness of Omega and Bounding the Dimension

Since $f(x) = L(x) - \Omega(x)$, $\theta(\Omega) = 0$. As we mentioned at the end of chapter 2, the hook integral is always non-negative; so Ω is a minimizer of the hook integral. Part of the hook integral looks like a logarithmic energy, for which there is a unique minimizer. A more general treatment of the following ideas can be found in the text "Logarithmic Potentials with External Fields" [3] by Saff and Totik.

4.1 Uniqueness of Omega

In [3], they begin by presenting the classical theory of logarithmic potentials. Let $M(\Sigma)$ denote all positive unit Borel measures, μ with support in a compact set, $\Sigma \subset \mathbb{C}$. The logarithmic energy of μ is defined as:

$$I(\mu) := -\iint \ln|z-t|d\mu(z)d\mu(t),$$

and the energy of Σ as:

$$V := \inf\{I(\mu) : \mu \in M(\Sigma)\}$$

It turns out that in the case when the energy V is finite, there exists a unique *equilibrium measure*, μ_Σ that achieves the minimum logarithmic energy.

An external field is then introduced in the form of a weight function, ω. In this weighted version it is no longer necessary to make Σ compact.

A weighted energy integral is then defined as:

$$\begin{aligned} I_\omega(\mu) &:= -\iint [\ln|z-t|\omega(z)\omega(t)]\, d\mu(z)d\mu(t) \\ &= -\iint \ln|z-t|d\mu(z)d\mu(t) + 2\int Q d\mu, \end{aligned}$$

where $Q(z) := -\ln\omega(z)$. Let $V_\omega = \inf\{I_\omega(\mu) : \mu \in M(\Sigma)\}$, be the minimum energy. The key theorem from [3] says that V_ω is finite and there is a unique minimizer, μ_ω such that $I_\omega(\mu_\omega) = V_\omega$. It also says the logarithmic energy of the minimizer is finite:

$$-\infty < -\iint \ln|z-t|d\mu_\omega(z)d\mu_\omega(t) < \infty.$$

One of the ways we were able to express the hook integral in the previous chapter was:

$$\theta(L) = C - 2\iint_S \ln|s-t|L'(s)L'(t)dsdt - 2\int_a^b [\Phi_1(b-t) + \Phi_1(a-t)]\, L'(t)dt,$$

where $C = 1 + 2\iint_S \ln|s-t|dsdt$ is a constant. Setting $\mu := L$ and $Q := -[\Phi_1(b-t) + \Phi_1(a-t)]$, we have that $I_\omega(L) = \theta(L) - C$. Therefore, the uniqueness of the minimizer of I_ω implies that θ has a unique minimizer as well.

We will not prove the entire theorem mentioned above, but we will show how the proof of the uniqueness is a consequence of the following lemma, also from [3]:

Lemma 4.1.1 *Let $\mu = \mu_1 - \mu_2$ be a signed Borel measure with compact support and total mass $\mu(\mathbb{C}) = 0$. If both of the positive measures μ_1 and μ_2 have finite logarithmic energy, then the logarithmic energy of μ is non-negative:*

$$I(\mu) := -\iint \ln|z-t|d\mu(z)d\mu(t) \geq 0,$$

and is zero if and only if $\mu = 0$.

We have already shown that $\Omega =: \mu_\omega$ is a minimizer. Suppose there is another minimizer $\bar{\mu}$. We set $\nu = \frac{1}{2}(\mu_\omega - \bar{\mu})$, which is a signed Borel measure with total mass zero since all of our allowable L's are normalized to have total mass 1. Further, since both μ_ω and $\bar{\mu}$ are minimizers, they each have finite logarithmic energy. Define

$$\begin{aligned} J &:= -\iint [\ln|z-t|\omega(z)\omega(t)]\, d\nu(z)d\nu(t) \\ &= -\iint \ln|z-t| d\nu(z)d\nu(t). \end{aligned}$$

The lemma tells us the $J \geq 0$ with equality if and only if $\nu = 0$. However,

$$I_\omega\left(\frac{1}{2}\left[\mu_\omega + \bar{\mu}\right]\right) + J = \frac{1}{2}\left(I_\omega(\mu_\omega) + I_\omega(\bar{\mu})\right) = V_\omega.$$

Since V_ω is the minimum, $I_\omega(\frac{1}{2}(\mu_\omega + \bar{\mu})) \geq V_\omega$. So we have that $V_\omega + J \leq V_\omega$ which implies $J = 0$. Thus, $\nu = 0$ and $\mu_\omega = \bar{\mu}$. Therefore the minimizer is unique.

4.2 The Sobolev Norm and Bounds on the Dimension

The Sobolev norm of f is defined as:

$$\|f\|_\theta^2 := \iint \left(\frac{f(s) - f(t)}{s - t}\right)^2 dsdt. \tag{4.1}$$

The following equation is given as Lemma 4 and proven in Vershik and Kerov's [4] paper.

$$\theta(\Omega + f) = \frac{1}{2}\|f\|_\theta^2 + 4\int_{|s|>1} f(s)\cosh^{-1}|s| ds. \tag{4.2}$$

We will use this norm to obtain upper and lower bounds for the dimension of a specific λ. In order to do this, we need to consider a specific partition λ, this puts us into a discrete case where we have the function f_λ. Recall that $f_\lambda := L_\lambda - \Omega$.

Let $s_i = \frac{i}{2\sqrt{n}}$ for $i \in \mathbb{Z}$. This divides the real line into pieces of length $\Delta s = \frac{1}{2\sqrt{n}}$

over which the function L_λ is linear. We choose minimizers, s_i^* of $[f_\lambda'(s)]^2$ on the interval $[s_i, s_{i+1}]$. Note that the derivative of L, and consequently f also, is undefined at the endpoints that happen to be one of the ups, u_j or one of the downs, δ_j of the partition λ. When this occurs we must take our minimizer to be the infimum of $[f_\lambda'(s)]^2$ on the open interval (s_i, s_{i+1}). This gives us the following formula for any $s, t \in [s_i, s_{i+1}]$,

$$\left(\frac{f_\lambda(s) - f_\lambda(t)}{s - t}\right)^2 \geq [f_\lambda'(s_i^*)]^2.$$

For this discrete case, with f_λ, we can redefine this Sobolev norm, to be a sum over the union of squares $s, t \in [s_i, s_{i+1}], i \in \mathbb{Z}$. Using this definition we can sum the above equation over these squares to obtain:

$$\|f_\lambda\|^2 \geq \sum_i [f_\lambda'(s_i^*)]^2 (\Delta s)^2 = \frac{1}{2\sqrt{n}} \sum_i [f_\lambda'(s_i^*)]^2 \Delta s.$$

Since the integral in (4.2) is non-negative, we see that $\theta(L_\lambda) \geq \frac{1}{2}\|f_\lambda\|^2$. This gives us:

$$\theta(L_\lambda)\sqrt{n} \geq \frac{1}{2}\|f_\lambda\|^2\sqrt{n} \geq \frac{1}{4}\sum_i [f_\lambda'(s_i^*)]^2 \Delta s = \int_{-\infty}^{\infty} [f_\lambda'(s)]^2 ds - \epsilon,$$

where ϵ is the error from the Riemann approximation. Note that since the integrand is positive, integrating over a smaller region does not increase the right-hand side. From equations (2.7) and (3.5):

$$-\frac{\ln \mu_n(\lambda)}{\sqrt{n}} \geq \frac{1}{4}\int_{-1}^{1} |L_\lambda'(s) - \Omega'(s)|^2 ds.$$

Since $\Omega'(s) < 0$ ($\Omega'(s) > 0$) for $s < 0$ ($s > 0$) and $L'(s) = \pm 1$ almost everywhere, the minimizer of the integrand above is $|sign(s) - \Omega'(s)|$. We define a constant c_0:

$$2c_0 = \frac{1}{4}\int_{-1}^{1} |sign(s) - \Omega'(s)|^2 ds = \frac{2}{\pi^2}(\pi - 2) \approx 0.2313.$$

And have that $-\frac{\ln \mu_n(\lambda)}{\sqrt{n}} \geq 2c_0$. Applying the definition of the Plancherel measure we find that:

$$dim(\lambda) \leq \sqrt{n!}e^{-c_0\sqrt{n}}. \tag{4.3}$$

Since c_0 came from an absolute minimum, the above inequality is true for every $\lambda \vdash n$.

We return our attention to the discrete case, and consider $\lambda \vdash n$, for n finite. Recall that $\theta(L_\lambda)$ denotes a discrete double sum (plus error), rather than an integral.

We let $\text{Par}(n) = \{\lambda \vdash n\}$ and $p(n) = |\text{Par}(n)|$, the Euler-Hardy-Ramanujan formula says:

$$p(n) \approx \frac{1}{4n\sqrt{3}}e^{k\sqrt{n}},$$

where $k = \frac{2\pi}{\sqrt{6}}$. We define the set of essential partitions of n, to be:

$$E_n := \{\lambda \in \text{Par}(n) : \theta(L_\lambda) < \frac{k}{\sqrt{n}}\}.$$

Suppose we have a partition $\lambda \notin E_n$, then $\theta(L_\lambda) \geq \frac{k}{\sqrt{n}}$. Thus $\mu_n(\lambda) \leq e^{-k\sqrt{n}}$. Using the approximation above,

$$\mu_n(\text{Par}(n)\backslash E_n) \leq p(n)e^{-k\sqrt{n}} = \frac{1}{4n\sqrt{3}}.$$

So the measure of the set $\text{Par}(n)\backslash E_n \to 0$ as $n \to \infty$ and $\mu_n(E_n) \to 1$. As we take the limit, all of the measure lies within E_n; this is why we referred to them as *essential* partitions. Choosing a random partition of n, for n very large, will very likely result in getting a partition in E_n.

E_n was defined in terms of an upper bound on the $\theta(L_\lambda)$, we could equivalently describe the essential partitions in terms of a bound on the dimension as follows:

$$E_n = \{\lambda \in \text{Par}(n) : dim(\lambda) > \sqrt{n!}e^{-\pi\sqrt{\frac{n}{6}}}\}. \tag{4.4}$$

We give the following lower bound:

$$\lim_{n\to\infty} \mu_n\{\lambda : dim(\lambda) > \sqrt{n!}e^{-\pi\sqrt{\frac{n}{6}}}\} = 1. \tag{4.5}$$

Note that this is not a bound on the dimension of every single partition, since clearly there are two partitions with dimension 1 for every $n, \lambda = (n)$ and $\lambda = (1^n)$. The bound is with regards to the Plancherel measure and the essential partitions. It is in this sense that we refer to this bound as a bound on the dimension of a *typical* partition.

Appendix

In this appendix we present the proofs of equations (3.9) and (3.10):

$$\begin{aligned}\int_a^{-1} \Phi_0(t-s)\Omega'(t)dt &= \int_a^{-1} -\ln 2|t-s|(-1)dt \\ &= \int_{a-s}^{-(1+s)} -\Phi_0(y)dy \\ &= \Phi_1(a-s)+\Phi_1(s+1).\end{aligned}$$

$$\begin{aligned}\int_1^{b} \Phi_0(t-s)\Omega'(t)dt &= \int_1^{b} -\ln 2|t-s|(1)dt \\ &= \int_{1-s}^{b-s} \Phi_0(y)dy \\ &= \Phi_1(b-s)+\Phi_1(s-1).\end{aligned}$$

$$\begin{aligned}I(s) &= \int_a^{b} \Phi_0(t-s)\Omega'(t)dt \\ &= \Phi_1(a-s)+\Phi_1(b-s)+\Phi_1(s+1)+\Phi_1(s-1)+I_1(s)\end{aligned}$$

where $I_1(s) := \int_{-1}^{1} \Phi_0(t-s)\Omega'(t)dt$. In order to find $I_1(s)$, we will need to use the following function $v_s(t)$, defined below:

$$v_s(t) := \int \Phi_0(t-s)dt = (s-t)\ln 2|t-s| + t.$$

$$\begin{aligned}I_1(s) &= \int_{-1}^{1} \Phi_0(t-s)\Omega'(t)dt \\ &= \Omega'(t)v_s(t)\Big|_{-1}^{1} - \frac{2}{\pi}\int_{-1}^{1} \frac{v_s(t)}{\sqrt{1-t^2}}dt \\ &= v_s(1)+v_s(-1)+I_2(s).\end{aligned}$$

A simple calculation shows that,

$$v_s(1) + v_s(-1) + \Phi_1(s+1) + \Phi_1(s-1) = 2s.$$

And thus we have the following formula for $I(s)$:

$$I(s) = \Phi(a-s) + \Phi(b-s) + 2s + I_2(s)$$

We now write $I_2(s)$ as:

$$\begin{aligned} I_2(s) &= \frac{2}{\pi}\int_{-1}^{1}\frac{(t-s)\ln 2|t-s|}{\sqrt{1-t^2}}dt - \frac{2}{\pi}\int_{-1}^{1}\frac{t}{\sqrt{1-t^2}}dt \\ &= \frac{2\ln 2}{\pi}\int_{-1}^{1}\frac{(t-s)}{\sqrt{1-t^2}}dt + \frac{2}{\pi}\int_{-1}^{1}\frac{(t-s)\ln|t-s|}{\sqrt{1-t^2}}dt \\ &= -\frac{2s\ln 2}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}dt + \frac{2}{\pi}\int_{-1}^{1}\frac{(t-s)\ln|t-s|}{\sqrt{1-t^2}}dt \\ &= -2s\ln 2 + \frac{2}{\pi}\int_{-1}^{1}\frac{t\ln|t-s|}{\sqrt{1-t^2}}dt - \frac{2s}{\pi}\int_{-1}^{1}\frac{\ln|t-s|}{\sqrt{1-t^2}}dt. \end{aligned} \tag{4.6}$$

We will now work with the two integrals above, in three seperate cases;
(i) $s < -1$
(ii) $s \in [-1, 1]$
(iii) $s > 1$
For convenience, we will combine (i) and (iii) above and write $\pm s > 1$.

In order to calculate the second integral above, we define the complex integral:

$$h(z) = \int_{-1}^{1}\frac{\ln(z-t)}{\sqrt{1-t^2}}dt.$$

After calculating boundary values of this function, we see that $\forall s \in [a, b]$:

$$h_+(s) + h_-(s) = 2\int_{-1}^{1}\frac{\ln|s-t|}{\sqrt{1-t^2}}dt.$$

Which closely resembles the second integral above.

Assuming $z \notin \mathbb{R}$, we use contour integration and the residue theorem, to write the derivative of $h(z)$ as follows:

$$h'(z) = \int_{-1}^{1} \frac{1}{(z-t)\sqrt{1-t^2}}dt = \frac{\pi}{\sqrt{z-1}\sqrt{z+1}}.$$

Notice that here the contour at infinity goes to zero.

Using the fundamental theorem of calculus:

$$h_+(s) + h_-(s) = h_+(-1) + h_-(-1) + \int_{-1}^{s} [h'_+(z) + h'_-(z)]dz.$$

We now consider the derivative evaluated at $s \in \mathbb{R}$ for each of our cases:
For $s \in [-1, 1]$,

$$h'_+(s) = -h'_-(s) \implies h_+(s) + h_-(s) = -2\pi \ln 2, \ \forall s \in [-1, 1].$$

The last equality above is a result of the fact that, for $s = 0$:

$$h_+(0) + h_-(0) = 2\int_{-1}^{1} \frac{\ln|t|}{\sqrt{1-t^2}}dt = -2\pi \ln 2.$$

For $\pm s > 1$,

$$\begin{aligned} h_+(s) + h_-(s) &= -2\pi \ln 2 \pm 2\pi \int_{\pm 1}^{s} \frac{1}{\sqrt{z^2-1}}dz \\ &= -2\pi \ln 2 \pm 2\pi \int_{0}^{\cosh^{-1}(\pm s)} \pm 1 dx \\ &= -2\pi \ln 2 + 2\pi \cosh^{-1}|s|. \end{aligned}$$

This leads us to the following formula for the second integral in equation (4.6),

$$-\frac{2s}{\pi}\int_{-1}^{1}\frac{\ln|t-s|}{\sqrt{1-t^2}}dt = 2s\ln 2 + \begin{cases} -2s\cosh^{-1}|s| & \pm s > 1 \\ 0 & |s| < 1. \end{cases}$$

We follow a similar approach to calculate the first integral in equation (4.6). Define another complex integral:

$$g(z) = \int_{-1}^{1}\frac{t\ln(z-t)}{\sqrt{1-t^2}}dt.$$

As before, calculating boundary values gives us that $\forall s \in [a,b]$:

$$g_+(s) + g_-(s) = 2\int_{-1}^{1}\frac{t\ln|s-t|}{\sqrt{1-t^2}}dt.$$

Assuming $z \notin \mathbb{R}$, using contour integration and the residue theorem we follow a similar approach as before. Although this time the contour at infinity does not tend toward zero, we use a first order approximation to obtain the following formula for the derivative of $g(z)$:

$$\begin{aligned} g'(z) &= \int_{-1}^{1}\frac{t}{(z-t)\sqrt{1-t^2}}dt \\ &= \frac{\pi z}{\sqrt{z-1}\sqrt{z+1}} - \pi. \end{aligned}$$

Again using the fundamental theorem;

$$g_+(s) + g_-(s) = g_+(-1) + g_-(-1) + \int_{-1}^{s}[g'_+(z) + g'_-(z)]dz,$$

We now consider the derivative of g evaluated at $s \in [-1,1]$,

$$g'_+(s) + g'_-(s) = -2\pi.$$

For $s = 0$,

$$2\int_{-1}^{1} \frac{t \ln|t|}{\sqrt{t^2-1}} dt = 0,$$

so $g_+(-1) + g_-(-1) = 2\pi$. So for $|s| < 1$, we have,

$$g_+(s) + g_-(s) = 2\pi + \int_{-1}^{s} -2\pi dz = -2\pi s.$$

For $\pm s > 1$,

$$\begin{aligned} g_+(s) + g_-(s) &= g_+(-1) + g_-(-1) + \int_{-1}^{s} \frac{\pm 2\pi z}{\sqrt{z^2-1}} dz + \int_{-1}^{s} -2\pi dz \\ &= 2\pi \pm 2\pi\sqrt{s^2-1} - 2\pi s - 2\pi. \end{aligned}$$

We arrive at the following formula for the first integral in equation (4.6),

$$\frac{2}{\pi}\int_{-1}^{1} \frac{t \ln|t-s|}{\sqrt{1-t^2}} dt = -2s + \begin{cases} \pm 2\sqrt{s^2-1} & \pm s > 1 \\ 0 & |s| < 1. \end{cases}$$

Now, substituting back into equation (4.6),

$$\begin{aligned} I_2(s) &= -2s\ln 2 + \frac{2}{\pi}\int_{-1}^{1} \frac{t \ln|t-s|}{\sqrt{1-t^2}} dt - \frac{2s}{\pi}\int_{-1}^{1} \frac{\ln|t-s|}{\sqrt{1-t^2}} dt \\ &= -2s + \begin{cases} -2s\cosh^{-1}|s| \pm 2\sqrt{s^2-1} & \pm s > 1 \\ 0 & |s| < 1. \end{cases} \end{aligned}$$

This completes the proof of equation (3.9).

Equation (3.10) can be shown in a similar fashion with the help of the following integral:

$$\int_{-1}^{1} \Phi_2(s)\Omega''(s) ds = \frac{1}{2}.$$

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