

Research Statement

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My primary research is in the field of geometric analysis. More specifically, I work in the areas of variational methods on piecewise flat manifolds, discrete conformal geometry, and circle patterns on triangulations.

Through the use of energy functionals and conformal invariants on triangulations I study discrete analogs of conformal structures, Einstein metrics, and constant scalar curvature metrics. As will be explained in this research summary, I have established the local behavior of discrete (normalized) Einstein-Hilbert functionals at Einstein metrics on a collection of triangulations of \mathbb{S}^3 .

Another area of my research concerns weighted Delaunay conformal variations on two dimensional triangulations. I have proven that the conformal parameters under such variations are unbounded when approaching a degeneration. This research is a component of a larger research program directed towards the development of a uniformization theorem for general discrete conformal structures.

I also have research interests in the study of angle structures on simplices and triangulations. I have proven a characterization theorem for fully general ‘hyper-ideal’ hyperbolic simplices using a generalization of the notion of dihedral angle. For two dimensional triangulated manifolds, I have established the existence of a global metric structure for a specific angle structure derived from the conformal class.

1 Discrete Einstein metrics and the Einstein-Hilbert-Regge functional

The well studied Einstein-Hilbert functional $\mathcal{E}\mathcal{H}$, and its normalization $\mathcal{N}\mathcal{E}\mathcal{H}$ on closed Riemannian manifolds are given by

$$\mathcal{E}\mathcal{H}(M, g) = \int_M R_g dV_g, \quad \mathcal{N}\mathcal{E}\mathcal{H}(M^n, g) = \frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{(n-2)/n}}.$$

In particular, critical points of $\mathcal{E}\mathcal{H}$ are Ricci-flat metrics, and critical points of $\mathcal{N}\mathcal{E}\mathcal{H}$ are Einstein metrics (see [2]). Einstein manifolds are of particular interest because Einstein metrics provide a highly symmetric or “best” geometry for the manifold.

T. Regge [13] introduced a functional defined on piecewise flat manifolds that is analogous to $\mathcal{E}\mathcal{H}$. We call this functional the *Einstein-Hilbert-Regge functional* and denote it by $\mathcal{E}\mathcal{H}\mathcal{R}$. Let $\mathcal{T} = (V, E, F, T)$ be a triangulation of a three dimensional manifold with vertices, edges, faces, and tetrahedra given by the sets V, E, F, T , respectively. For a piecewise flat three-manifold, denote by $\beta_{e<t}$ the *dihedral angle* at the edge e within the tetrahedron t . The *edge curvature* at the edge e is given by

$$K_e = \left(2\pi - \sum_{t \in \mathcal{T}} \beta_{e<t}\right) \ell_e,$$

where ℓ_e is the edge length. This definition of edge curvature is the curvature measure appearing in [6] evaluated at the edge e . The $\mathcal{E}\mathcal{H}\mathcal{R}$ functional is given by

$$\mathcal{E}\mathcal{H}\mathcal{R} = \sum_{e \in E} K_e.$$

We turn now to my research, to obtain a functional that is invariant under uniform scalings of the edge lengths we normalize $\mathcal{E}\mathcal{H}\mathcal{R}$. The *length normalized Einstein-Hilbert-Regge functional*, $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$, and *volume normalized Einstein-Hilbert-Regge functional*, $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$, are given by

$$\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R} = \frac{\mathcal{E}\mathcal{H}\mathcal{R}}{L}, \quad \mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R} = \frac{\mathcal{E}\mathcal{H}\mathcal{R}}{V^{1/3}},$$

where \mathcal{L} is the sum of all edge lengths, and \mathcal{V} is the total volume summed over all tetrahedra.

The volume normalization is a natural analogue to $\mathcal{N}\mathcal{E}\mathcal{H}$, however the formula for volume of a Euclidean simplex as a function of edge lengths is quite complicated. This makes analysis of $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ difficult even on the simplest triangulations. For this reason we consider a normalization that is linear in the edge lengths.

The critical points of the $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ and $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ functionals give two notions of Einstein metrics. A metric is an \mathcal{L} -Einstein metric if there exists $\lambda_{\mathcal{L}} \in \mathbb{R}$ such that for all edges e ,

$$K_e = \lambda_{\mathcal{L}} \ell_e.$$

Here $\lambda_{\mathcal{L}} = \mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$. The metric is a \mathcal{V} -Einstein metric if there exists $\lambda_{\mathcal{V}} \in \mathbb{R}$ such that for all $e \in E$,

$$K_e = \lambda_{\mathcal{V}} V_e.$$

Here $\lambda_{\mathcal{V}} = \frac{\mathcal{E}\mathcal{H}\mathcal{R}}{3\mathcal{V}}$, and $V_e = \ell_e \frac{\partial \mathcal{V}}{\partial \ell_e}$.

The double tetrahedron and pentachoron are two small triangulations of \mathbb{S}^3 that we hope will lead to a more general analysis of $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ and $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$. The double tetrahedron consists of two congruent tetrahedra glued along corresponding faces. The pentachoron has the combinatorics of the boundary of the four-simplex.

Theorem 1 *The following hold for the double tetrahedron and the pentachoron:*

1. *Equal lengths metrics are \mathcal{L} -Einstein and \mathcal{V} -Einstein.*
2. *$\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ is positive semi-definite at equal length metrics.*
3. *$\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ has mixed signature at equal length metrics.*

The results in Theorem 1 pertaining to the double tetrahedron are proven in [4], while the results on the pentachoron are proven in [3].

The Hessian matrix for either $\mathcal{L}\mathcal{E}\mathcal{H}\mathcal{R}$ or $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ on these triangulations has a centrosymmetric structure. This enables one to determine the eigenvectors and eigenvalues and show that they have a structure related to the combinatorics of the triangulation. Furthermore, I have proven that the equal length metrics are the unique \mathcal{L} -Einstein metrics on the double tetrahedron.

Some of the eigenvectors of the Hessian matrix for the pentachoron have the same structure as the eigenvectors for the double tetrahedron. A next step in my research is to utilize this property to generalize some of these results to a collection of highly symmetric triangulations.

We have some knowledge of the global behavior of the $\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ functional on all triangulations. Indeed in [3] I prove the following.

Theorem 2 *$\mathcal{V}\mathcal{E}\mathcal{H}\mathcal{R}$ is unbounded above on all triangulated three-manifolds.*

This result follows from an analysis of a particular family of admissible metrics that approach a degeneration wherein the volumes of all tetrahedra approach zero. The combinatorial manifold conditions inherent to the triangulation are used to secure the result.

Describing the global and local behavior of normalized Einstein-Hilbert-Regge functionals for all triangulated three-manifolds is at the center of my current research. We have found triangulations of \mathbb{S}^3 where the behavior is quite different than that of the double tetrahedron and pentachoron. Additionally, the normalized functionals do not preserve their signature even on the double tetrahedron and pentachoron. I have developed a general theory of normalizations of the $\mathcal{E}\mathcal{H}\mathcal{R}$ functional to approach this problem in [3]. Finding a normalization whose Hessian can be efficiently studied is another primary research goal of mine.

A discrete conformal structure can be used to determine the edge lengths as a function of vertex weights. As developed in [8], one can develop a notion of constant scalar curvature as the critical points of a normalized $\mathcal{E}\mathcal{H}\mathcal{R}$ functional under conformal variations, see Section 2 and [8] for details. In [4] we have proven the existence of constant scalar curvature metrics on the double tetrahedron in all conformal classes. Furthermore, we show in [4] that equihedral metrics on the double tetrahedron (wherein opposite edges have equal length) are constant scalar curvature metrics. In [5] we provide extensions of our results on constant scalar curvature metrics to other highly symmetric triangulations of \mathbb{S}^3 (cyclic polytopes). My current research in this area involves developing a better understanding of constant scalar curvature metrics on general triangulations.

2 Towards a discrete uniformization theorem

The well known uniformization theorem for smooth Riemannian surfaces states that there is a metric with constant Gaussian curvature in any given conformal class. Successes in the development of a discrete uniformization theorem have come about recently from the study of circle packing metrics. Conjectured in a talk by Thurston in 1985 [16], and proven by Rodin and Sullivan in 1987 [16], circle packing metrics were shown to approximate conformal maps. Discrete Riemann mapping theorems utilizing circle packing metrics show that any simply connected triangulation can be mapped to a circle packing metric of the disk with externally tangent circles. The resultant circle packing of the disk produces a triangulation with constant zero curvature.

Using a generalized notion of conformal structure on a piecewise flat triangulation that appears in [8], the lengths of the edges are calculated from vertex weights $\{f_i\}_{i \in V}$ and edge weights $\{\eta_{ij}\}_{ij \in E}$ by the formula

$$\ell_{ij}^2 = e^{2f_i} + e^{2f_j} + 2e^{f_i+f_j}\eta_{ij}. \quad (1)$$

A conformal class is given by fixing a collection of edge weights $\{\eta_{ij}\}_{ij \in E}$. Circle packing metrics constitute a single conformal class given by $\eta_{ij} = 1$ for all $ij \in E$. General conformal classes lose some of the properties and simplicity of circle packing metrics. In particular, all circle packing metrics automatically produce weighted Delaunay triangulations, however this is not the case for all discrete conformal classes. Consequently, the study of conformal variations on general conformal structures must address these issues.

Let \vec{E} denote that set of ordered edges consisting of ordered pairs (i, j) where $ij \in E$. For $(i, j) \in \vec{E}$, the *premetric* is given by the set of quantities $\{d_{ij}\}_{(i,j) \in \vec{E}}$ given by the formula

$$d_{ij} = \frac{\ell_{ij}^2 + e^{2f_i} - e^{2f_j}}{2\ell_{ij}}. \quad (2)$$

The premetric is related to the edge lengths according to

$$\ell_{ij} = d_{ij} + d_{ji}. \quad (3)$$

Equations (2) and (3) give a partitioning of the edge lengths of a triangle so that the perpendiculars of the three edges define a geometric center of the triangle. Let the *edge height* $h_{ij,k}$ of the edge ij in the triangle $\{i, j, k\}$ is given by the formula

$$h_{ij,k} = \frac{d_{ik} - d_{ij} \cos \gamma_{i,jk}}{\sin \gamma_{i,jk}},$$

where $\gamma_{i,jk}$ is the face angle of $\{i, j, k\}$ at vertex i . The weighted Delaunay condition for a pair of contiguous triangles $\{i, j, k\}, \{i, j, l\}$ is given by

$$h_{ij,k} + h_{ij,l} > 0.$$

My research in the area of discrete uniformization focuses on boundary behavior under discrete conformal variations. The boundary of the space of admissible metrics consist of metrics with degenerate triangles, that is, triangle inequality violations. In a collaboration with D. Glickenstein, we are developing conformal variations that approach constant curvature metrics that are guided by a system of differential equations or an energy functional. Hence, understanding the behavior of discrete conformal variations near triangle degenerations is absolutely necessary for the development of such a result. My primary result in this research program is the following that appears in [3].

Theorem 3 *For any degenerating conformal variation of a triangulation that remains weighted Delaunay, there exist $f_i \rightarrow \pm\infty$ as the variation approaches the degeneration.*

3 Angle structures on hyperbolic n -simplices

Simplices of any geometry and dimension can be conformally embedded within Euclidean space of possibly higher dimension. Hyperbolic n -simplices are the intersection of $n+1$ half-spaces in a conformal model of \mathbb{H}^n

(e.g. within the hyperboloid model of \mathbb{H}^n inside \mathbb{E}^{n+1} with the Lorentzian inner product). The intersection of half-spaces creates the codimension-two faces of the simplex wherein an interior dihedral angle is formed. The dihedral angles at the codimension-two faces can be easily computed by an appropriate bilinear form on the normal vectors to the codimension-one hyperplanes that define the half-spaces.

Consider a set of positive numbers $\Theta = \{\theta_{ij}\}_{i,j=0}^n$ satisfying $\theta_{ij} = \theta_{ji}$ and $\theta_{ii} = \pi$. For Θ to be the dihedral angles of a spherical, Euclidean, or hyperbolic simplex, several nonlinear conditions and inequalities must be satisfied. The conditions on Θ are well studied in the case of these classical geometric simplices, see [7],[11], [12]. More recently, a class of generalized hyperbolic simplices have been studied in a similar setting wherein the vertices of the simplex are *hyper-ideal*, lying beyond the sphere at infinity in hyperbolic space [17]. This work requires that all codimension-two faces of the simplex intersect hyperbolic space forming a dihedral angle. My research in this area utilizes inversive distances that generalize the notion of dihedral angle allowing for the removal of any conditions on the codimension-two faces. Inversive distances have a geometric formulation using the hyperplanes defining the codimension-one faces of a simplex and are invariant under isometries of hyperbolic space.

For an n -simplex σ^n , with vertices $\{v_1, \dots, v_{n+1}\}$, let the ij^{th} -face of σ^n , for $i \neq j$, be the codimension-two face given by the vertices $\{v_1, \dots, v_{n+1}\} \setminus \{v_i, v_j\}$. We associate to each ij^{th} -face of σ^n a real number $\eta_{ij} \in [-1, \infty)$ called the *inversive distance*. When $\eta_{ij} \in [-1, 1]$, the *dihedral angle* θ_{ij} and η_{ij} of the ij^{th} -face are related by the expression:

$$\eta_{ij} = \cos(\theta_{ij}). \quad (4)$$

We formally set $\eta_{ii} = -1$ for all $i \in \{1, 2, \dots, n+1\}$. The Gram matrix G of σ^n is the real symmetric unidiagonal matrix given by

$$G_{ij} = -\eta_{ij}.$$

Codimension-one hyperplanes in the ball model of n -dimensional hyperbolic space are given by an $(n-1)$ -sphere intersecting the unit sphere orthogonally. Given two such codimension-one hyperplanes Π_i, Π_j , in the ball model of hyperbolic space, let r_i, r_j be the radii of these spheres and let ℓ_{ij} be the distance between the centers as measured in the ambient n -dimensional Euclidean space. The inversive distance between these hyperplanes is given by

$$\eta_{ij} = \frac{\ell_{ij}^2 - r_i^2 - r_j^2}{2r_i r_j}.$$

This expression is invariant under isometries of hyperbolic space and is equivalent to Equation (4) if the hyperplanes intersect.

In [3] I prove the following characterization result for hyper-ideal simplices.

Theorem 4 *Given a set of positive numbers*

$$\{\eta_{ij} : \eta_{ii} = -1, \eta_{ij} = \eta_{ji} \in [-1, \infty), \text{ for } i, j \in \{1, \dots, n+1\}\},$$

there exists a hyperideal n -simplex in \mathbb{H}^n with inversive distances at the ij^{th} -face equal to η_{ij} iff the Gram matrix G satisfies the following

1. *The signature of G is $(n, 1)$ (n positive and one negative eigenvalue)*
2. *$\det(G) < \text{cof}_G(i, i) < 0$, for $1 \leq i \leq n+1$.*

The proof of Theorem 4 roughly follows the techniques of [11] and [17] using inversive distances to address the unique complications that arise when considering generalized hyperideal simplices. Furthermore, in this work we provide additional details pertaining to the structure of the compact truncation of the hyperideal simplex.

My interest in this area of research extends into the field of discrete conformal geometry and weighted piecewise flat triangulations. A simplex with vertex weights in a weighted piecewise flat triangulation forms a *Möbius simplex* with inversive distances (dihedral angle) at the ij^{th} -faces given by the intersection angle formed by the weight spheres at vertices i and j . When the vertex weights are small compared to the edge

lengths of the piecewise flat triangulation, the Möbius simplices have the structure of the hyperideal simplices characterized in Theorem 4.

In the study of general discrete conformal structures, a conformal class is given by a set of inversive distances. My current research in this area involves the study of conformal classes that satisfy the conditions on the inversive distances given by Theorem 4 or one of the other characterization theorems for classical geometric simplices. The conformal classes that satisfy these conditions at each simplex of a triangulation are candidates for the proper conditions on the inversive distances. I am researching how these conformal classes behave under conformal variations and how the Möbius simplices relate to ‘ideal’ metrics obtained from such conformal variations. Furthermore, in dimension two the Möbius simplices give an alternative angle structure to the triangulation that has interesting global metric properties. The next section is details this area of my research.

4 Angle structures on triangulated surfaces

An angle structure \mathcal{A} on a triangulated surface consists of an assignment of positive face angles $\alpha_{i,jk}, \alpha_{j,ik}, \alpha_{k,ij}$ to each triangle $\{i, j, k\} \in F$. A homogeneous angle structure \mathcal{A} satisfies one of the following:

1. If for all $\{i, j, k\} \in F$ we have $\alpha_{i,jk} + \alpha_{j,ik} + \alpha_{k,ij} = \pi$, we call the angle structure \mathcal{A} *Euclidean*.
2. If for all $\{i, j, k\} \in F$ we have $\alpha_{i,jk} + \alpha_{j,ik} + \alpha_{k,ij} < \pi$, we call the angle structure \mathcal{A} *hyperbolic*.
3. If for all $\{i, j, k\} \in F$ we have the following inequalities

$$\begin{aligned} \alpha_{i,jk} + \alpha_{j,ik} + \alpha_{k,ij} &> \pi, \\ \alpha_{i,jk} + \alpha_{j,ik} - \alpha_{k,ij} &< \pi, \\ \alpha_{i,jk} - \alpha_{j,ik} + \alpha_{k,ij} &< \pi, \\ -\alpha_{i,jk} + \alpha_{j,ik} + \alpha_{k,ij} &< \pi, \end{aligned}$$

we call the angle structure \mathcal{A} *spherical*.

For a Euclidean angle structure, each triangle can be realized as a geometric triangle with angles given by the angle structure. For a choice of scaling, let $L(e, t)$ be the length of the edge e as calculated within the triangle t . A global metric on these geometric triangles consists of a scaling for each triangle such that

$$L(e, t) = L(e, t'),$$

for any pair of triangles t, t' containing the edge e .

Angle structures do not guarantee the existence of a global metric. Finding angle structures that permit a global metric is the core problem of this field of research. Given a set of edge weights $\{I_{ij}\}_{ij \in E}$, I. Rivin in [14] used a variational method to find a Euclidean angle structure satisfying

$$\alpha_{k,ij} + \alpha_{l,ij} = I_{ij},$$

that admits a global metric. Similar results and methods appear in [18],[10],[9], and [15] for Euclidean, spherical and hyperbolic angle structures. In my work, I use edge weights to determine a symmetric type of angle structure and then show that a global metric exists.

Given a set of edge weights $\{\eta_{ij}\}_{ij \in E}$, the angle structure for a triangle $\{i, j, k\}$ is given by

$$\cos(\alpha_{i,jk}) = \eta_{jk}, \quad \cos(\alpha_{j,ik}) = \eta_{ik}, \quad \cos(\alpha_{k,ij}) = \eta_{ij}. \quad (5)$$

We will call the angle structure given by (5) the *Möbius angle structure* corresponding to the set of edge weights. We will say that a set of edge weights *Euclidean*, *hyperbolic*, or *spherical* provided that the corresponding Möbius angle structure is Euclidean, hyperbolic, or spherical respectively.

Notice that for a pair of triangles $\{i, j, k\}, \{i, j, l\}$ sharing the edge $\{i, j\}$ we have the fundamental symmetry

$$\alpha_{k,ij} = \arccos(\eta_{ij}) = \alpha_{l,ij}. \quad (6)$$

A *dual cycle* is a sequence of faces $\vec{f} = (f_0, f_1, \dots, f_{n-1}, f_n = f_0)$ such that each pair of consecutive faces f_{i-1}, f_i share an edge e_i . The *holonomy* of a dual cycle is given by the quantity

$$\text{hol}(\vec{f}) = \prod_{i=0}^{N-1} \frac{\sin(\theta_i^+)}{\sin(\theta_i^-)},$$

where θ_i^+ is the angle of face f_i opposite edge e_{i+1} , and θ_i^- is the angle of face f_i opposite edge e_i .

For Euclidean angle structures let $L_{\mathbb{E}}(e, f)$ give a normalized Euclidean length of the edge e within the face f computed from the angle structure (any normalization will do, for example, normalize the triangles so that they are circumscribed by a unit radius circle). If $\text{hol}(\vec{f}) = 1$, the faces $\{f_i\}_{i=0}^{N-1}$ of the dual cycle can be scaled consistently by positive numbers $\{\rho_i\}_{i=0}^{N-1}$ so that for $0 \leq i \leq N-1$ we have

$$\rho_i \cdot L_{\mathbb{E}}(e_{i+1}, f_i) = \rho_{i+1} \cdot L_{\mathbb{E}}(e_{i+1}, f_{i+1}), \quad (7)$$

For spherical and hyperbolic angle structures, $\text{hol}(\vec{f})$ measures a similar property. Let $\mathbb{K} = \mathbb{S}^2$ or \mathbb{H}^2 with curvature $k = 1$ or $k = -1$ respectively. Define $L_{\mathbb{K}}(e, f)$ to be the length of edge e within face f as computed from the angle structure using the spherical or hyperbolic cosine law. If $\text{hol}(\vec{f}) = 1$ the spherical (hyperbolic) triangles $\{f_i\}_{i=0}^{N-1}$ can be scaled consistently by positive numbers $\{\rho_i\}_{i=0}^{N-1}$ so that for $0 \leq i \leq N-1$ we have

$$\rho_i \cdot \sin(\sqrt{k} \cdot L_{\mathbb{K}}(e_{i+1}, f_i)) = \rho_{i+1} \cdot \sin(\sqrt{k} \cdot L_{\mathbb{K}}(e_{i+1}, f_{i+1})). \quad (8)$$

Utilizing the symmetry of the angle structure expressed in Equation (6) as well as the Euclidean, spherical, or hyperbolic law of sines It is shown in [3] that the holonomy of all dual cycles is trivial for Euclidean, spherical, and hyperbolic sets of edge weights. In [3] I prove the following existence result for global metrics.

Theorem 5 *For any Euclidean, spherical, or hyperbolic set of edge weights on a two-dimensional triangulation, the associated Möbius angle structure admits a global metric in the sense of either Equation (7) or (8).*

In the case of spherical and hyperbolic angle structures, it is possible to model Equation (8) geometrically. If we interpret the scaling constants ρ_i, ρ_{i+1} as spherical (hyperbolic) radii for the triangles f_i, f_{i+1} respectively, then Equation (8) is equivalent to having specific orthogonal projections of the shared edge be the same length.

My current research in this area includes extending Theorem (5) to ideal hyperbolic angle structures and hyperideal angle structures. Indeed, if we interpret the edge weights $\{\eta_{ij}\}_{ij \in E}$ as the conformal class constants and use the radii and inversive distances conformal structure given by Equation (1), then we have the following:

1. Euclidean, spherical, and hyperbolic edge weights correspond to generalized circle packing metrics with intersecting weight circles.
2. Ideal hyperbolic edge weights correspond to circle packing metrics.
3. Hyperideal hyperbolic edge weights correspond to generalized circle packing metrics with non-intersecting weight circles.

Further development of the geometric relationship between Möbius angle structures and the corresponding discrete conformal structures summarizes my current research program in this area.

My current research also includes the study of Möbius angle structures on three dimensional triangulations. It is possible to formulate analogs of Equations (7) and (8) for three dimensional Möbius angle structures utilizing scalings of the link simplex (spherical geometry).

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