Math 223 Super Challenge

October 19

The problems given below are more difficult than the problems found in the exams and on your homework. To complete these problems will often require an extended amount of time and possibly some research to find a solution. You are required to complete **two** of these "super-challenge" problems by November 30th. You are **not** allowed to work in groups on these problems. I strongly encourage you to meet with me during office hours for guidance and further information on the problems you are working on.

1. Let
$$f(x,y) = \frac{2x^2 y}{x^4 + y^2}.$$
First, prove that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any and every straight line through the origin. Second, find a curve $y = g(x)$ passing through the origin such that $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along this curve. Finally, explain why the first two parts of this problem are not contradictory.

2. Consider the following function
$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$
Compute the first-order partial derivatives of this function for all points in the plane. Prove that f and its first-order partial derivatives are continuous at the origin. Finally, show that the mixed second-order partial derivatives are not equal at the origin.

3. Use triple integration to derive a formula for the volume of a general tetrahedron in terms of edge lengths.

4. Three companies: Axis Chemicals, Manchurian Global, Luthorcorp, produce products in quantities A, M, L respectively. The weekly profits to each are given by the following:
$$\begin{array}{cr} \text{Axis Chemicals:} & P_{AC} = 1000A - A^2 - 2AM, \\ \text{Manchurian Global:} & P_{MG} = 2000M - 2M^2 - 4ML, \\ \text{Luthorcorp:} & P_L = 1500L - 3L^2 - 6AL. \end{array}$$
If each firms acts independently to maximize it's weekly profit, what will those profits be? If Axis Chemicals and Luthorcorp join forces and decide to maximize their combined total profit, what effects will this have? Give a detailed answer to this problem and assume that it is known to Manchurian Global that Axis and Luthorcorp have joined forces.

5. Consider a set of N data points $\{(x_i, y_i)\}_{i=1}^N$. The least-squares line $y = mx + b$ is the line that minimizes the squares of the distances to each of the data points simultaneously. More precisely, define for $1 \leq i \leq N$:
$$d_i = y_i - (m \cdot x_i + b).$$
The parameters m, b of the least-squares line minimize the following quantity:
$$d_1^2 + d_2^2 + ... + d_N^2.$$
Show how to find the parameters m, b and use the second derivative test to show that they minimize the squares of the distances. Finally, provide an explicit example of your work with real data points (your choice of data).

6. Consider a homogeneous spherical ball of radius a centered at the origin with density δ. Show that the gravitational force F exerted by this ball on a point mass m located at the point $(0,0,c)$, where $c > a$ is the same as though all the mass of the ball were concentrated at its center $(0,0,0)$. Repeat the problem where the point mass m is replaced with another homogeneous spherical ball with radius a' and density δ'.

7. Consider a solid spherical ball of radius $\sqrt{3}$ with a 2-by-2 square hole cut symmetrically through the ball. Find the surface area, and volume of the object.

8. If C is the line segment from (x_1, y_1) to (x_2, y_2) show that

$$\int_C xdy - ydx = x_1 y_2 - x_2 y_1.$$

   Next, use this fact and Green's Theorem to find the area of the following:

   (a) A triangle with vertices $(0,0), (x_1, y_1), (x_2, y_2)$,
   (b) An equilateral triangle (inscribed in the unit circle),
   (c) A regular pentagon (inscribed in the unit circle),
   (d) A regular 17-gon (inscribed in the unit circle).

9. Read through section 15.3 in your text and complete problems 35,41,45.

10. Project 2 on page 922 of your textbook.

11. Prove the following identity

$$\nabla (F \cdot G) = (F \cdot \nabla) G + (G \cdot \nabla) F + F \times curl\,(G).$$

12. Find the positively oriented simple closed curve C for which the line integral below is maximized:

$$\int_C \left(y^3 - y\right) dx - 2x^3 dy.$$

13. Present a complete proof of the following theorems. Every step in the proofs should be completely described with no details left out.

    (a) Stokes' Theorem
    (b) Green's Theorem
    (c) Divergence Theorem

14. Invent an interesting problem and solve it. (You must have my approval if you are interested in this).

15. More to come...