Final Review

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1 Ossine region matching

1.1 Stokes flow

In stokes flow (for example around a sphere or cylinder) we find that in the cylinder case, the problem can't be solved in closed form using the BC's at hand. It's due to a non-uniformity when we throw away the acceleration terms in our equations. Using the following scales we obtain the still exact Stokes equation:

$$u = Uu', v = Vv', t = L/Ut', x = Lx', y = Ly'$$
$$p = p_\infty + \mu U/Lp'$$

we get

$$(R = UL/\nu)\frac{D\mathbf{u}'}{Dt'} = -\nabla' p' + \nabla^{2\,\prime}\mathbf{u}$$

For low Reynolds number we get stokes equation. There is a problem for large $r' \propto O(1/R)$, i.e. the acceleration term can no longer be ignored. For this case where we rescale the distance and velocity, to the following: $r = (a/R)\tilde{r}, \mathbf{u} = U(e_x + R\mathbf{u}'), p = p_\infty + (\mu U/L)R^2\tilde{p}$. We get the Osseen region equation:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial x} + R\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} = -\nabla \tilde{p} + \nabla^2 \tilde{\mathbf{u}}$$

The solution in case of the sphere for example produces a wake downstream of the body where velocity drops off at a rate of only $O(1/r)$.

1.2 Increasing Reynolds number

Lose of fore and aft symmetry. The following is based on diameter:

	Sphere	Cylinder
standing eddies	$Re > 6$	$Re > 24$
shedding	$Re > 30/40$	$Re > 130$

The shedding of vortices can still be seen for $Re \sim 5000 \to 20000$.

2 Vorticity dynamics

2.1 Vorticity kinematics

So some definitions: $\nabla \times \mathbf{u} = \vec{\omega}$, and via the Stokes theorem this implies that the circulation C through a material curve c,

$$C = \oint_{c'} \mathbf{u} \cdot d\mathbf{l} = \oint_A \nabla \times \mathbf{u} d\mathbf{A} = \oint \vec{\omega} d\mathbf{A}$$

A vortex tube follows the material curve flux as it travels through space at some instant in time.

Vortex stretching to maintain C and the area decreases/increases.

2.2 Vorticity dynamics

From the momentum equation we can take the curl and using some vector identities can get vorticity equation, i.e.

$$\frac{D\vec{\omega}}{Dt} = \underbrace{\vec{\omega} \cdot \nabla \mathbf{u}}_{\text{vortex stretching}} + \overbrace{\nabla \times \mathbf{f}}^{\text{force}} + \underbrace{\nu \nabla^2 \vec{\omega}}_{\text{diffusion}}$$

Can consider effect of vortex stretching by considering streamwise (to $\mathbf{u}$) and normal, i.e.

$$\vec{\omega} \cdot \mathbf{u} = \omega \frac{\partial u_t}{\partial s}\mathbf{e}_s + \omega \frac{\partial u_n}{\partial s}\mathbf{e}_n$$

Then one term stretches the vorticity and another rotates them. These two mechanisms maintain the conservation of circulation.

In 2D flow and unidirectional flow, there is no vortex stretching. In inviscid flow, VL is fluid material line.

We can obtain the time rate of change of the circulation of a material curve as it moves through space, it gives that

$$\frac{dC}{dt} = \oint_c \frac{\partial \mathbf{u}}{\partial t} d\mathbf{l} + \oint_c (d\mathbf{l} \cdot \mathbf{u})\mathbf{u}$$

Subsituting the momentum equation into the first integrand we can find out that the change in circulation only originates form diffusion (assuming single valued pressure, conservative force, and magnitude of velocity):

$$\frac{dC}{dt} = \nu \oint_c \nabla^2 \mathbf{u} d\mathbf{l}$$

We can also obtain the time rate of change of the mean-square vorticity in a fluid element,

$$\frac{d}{dt}\int_V \frac{1}{2}|\vec{\omega}|^2 dV = \int_V \omega_i \omega_j \frac{\partial u_i}{\partial x_j} dV - \nu \int_V \frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} dV + \int_{A(V)} \frac{\partial}{\partial x_j}(\frac{1}{2}\omega_i\omega_i) n_j dA$$

There is an apparent paradox, since circulation can only change over time due to viscous effects but mean square vorticity increases as a result of vortex stretching. A similar situation is that of conservation of momentum.

3 Boundary layer

3.1 Suction profile

The suction problem ($v = -V$ at $y = 0$) over a flat plate gives rise to an exponential profile for the velocity which depends only on y, $u = U(1 - \exp(Vy/\nu)$.

3.2 Laminar BL

In general, vorticity will diffuse in the y direction, and will be convected downstream in the case of a growing boundary layer.

In high Reynolds number flow ($Re = UL/\nu \gg 1$) due to scaling both distance scales with L_g (geometric length scale) then we obtain that

$$\frac{D\mathbf{u'}}{Dt'} = -\nabla' p' + \frac{1}{R}\nabla^2 \mathbf{u}'$$

The modified scaling for the BL equation is as follows:

$$x = L_g x', y = R^{-1/2} L_g y, v = R^{-1/2} U v', u = U u',$$
$$p = p_\infty + \rho U^2 p', t = L/U$$

It leads to the boundary layer equation:

$$\frac{Du'}{Dt'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial (y')^2}, \ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

Also we obtain from the y-momentum that $\partial p/\partial y = 0$ due to these scalings. The pressure gradient is set as forcing term that matches the pressure found from the inviscid solution.

The leading order effect of BL region on inviscid flow is to displace the effective surface from the actual surface to add displacement thickness.

From Bernoulli's $P/\rho + U^2/2 = Const$, $\partial p/\partial x + U\partial U/\partial x = 0$. So we can get the forcing term dP/dx from this. For free stream flow $K_1 x^m$, $0 < m < 1$ is acute angle. $m = 1$ we get stagnation flow, and $m > 1$ is flow into a corner. For $m < 0$, flow past an horizontal corner.

Characteristics of BL with varying pressure gradients:

- Favorable pressure gradient (pressure decreasing with downstream distance ($dP/dx < 0$)) $\Rightarrow$ profile is fuller, modest changes even for stronger accelerations.
- For adverse gradients ($dP/dx > 0$), the significant changes in profile.
- For $m = -0.091 \Rightarrow \partial u/\partial y|_{y=0} = 0$. Presumably the flow would separate.

There is an order $R^{-1/2}$ to the outer flow due to the boundary condition at the edge of the boundary layer, i.e. $\tilde{v} = O(R^{-1/2})$ produces the correction to the outer flow.

3.3 Laminar jet

- No free stream flow.
- Large reynolds number.
- Transverse diffusion is important, axial not so much.
- For $L_x \gg L_y$, $dp/dy = 0 \Rightarrow p = f(x)$. Since $BC: p \to p_\infty$ then $p = p_\infty \Rightarrow dp/dx = 0$.

Get equations for steady state:

$$uu_x + vu_y = \frac{1}{R} u_{yy}$$

do the next scaling as $y = 1 + R^{-1/2} y'$ and $v = R^{-1/2} v'$. We obtain the BL equation as previously formulated and also get the BC's:

(1) $u = 0, y \to \pm\infty$, (2) $v = 0, y = 0$, (3) $\partial u/\partial y = 0, y = 0$

and IC: $u = u_0(y), x = 0$. For this equation we can show that the momentum flux is a constant over the cross-section (we also use the continuity equation). This allows to introduce a similarity structure.

3.4 Wake flow

This is a flow we can linearize far from the body (in uniform flow) in the wake region. Suppose $u = U(1 - \epsilon u')$ and from continuity we can get the scale on v. Then we can get linearized equation assuming that only transverse diffusion is important. (Also $p = p_\infty$). We also assume $L_x \gg L_y$. We get equation (dimensional):

$$\frac{\partial u'}{\partial x} = \frac{\partial^2 u'}{\partial y^2}$$

and can show using continuity from the above equation that

$$Q = \int u dy$$

Volume flux can be shown to be constant with x. It allows for similirarity solution with BC's: $u' \to 0, y \to \infty$ and $du'/dy = 0$ and I suppose that $u' \to 0$ as $x \to 0$.