Analysis - Final Review of MATH527A/B
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1 Metric Spaces

Definition A metric d on a space M satisfies the following:

- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x), \; \forall \; x, y \in M$
- $d(x, z) \leq d(x, y) + d(y, z)$

with $d : M \times M \mapsto [0, \infty)$.

Example: $l^p(\mathbb{R}, \mathbb{N}$ with $||x||_p = (\sum_{\mathbb{N}} |x_i|^p)^(1/p)$. In particular $l^1 \subset l^2 \dots l^\infty$. Also for $1 \geq p < q \geq \infty$, $l^q \subset l^p$.

Convergence in metric space: If $x_n \to x$ in (M, d) then $\forall \epsilon > 0, \; \exists N_\epsilon$ such that

$$n > N_\epsilon \to d(x_n, x) < \epsilon$$

Cauchy convergence A sequence x_n is cauchy iff the following holds: $\forall \epsilon > 0$ there exists N_ϵ such that

$$n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon$$

In particular if all Cauchy sequences converge in the space M then the space is complete. ALso all converging sequences are Cauchy.

Coarseness If $x_n \to x$ in (M, d) implies that $x_n \to x$ in (M, d') then d is coarser than d'.

Equivalence of metrics: If two metrics are equivalent then

$$x_n \to x \text{ in } (M, d) \Leftrightarrow x_n \to x \text{ in } (M, d')$$

E.g. On $\mathbb{R}^n$ all l^p norms are equivalent. An alternate characterization is the following: if given x and $\epsilon > 0$, $\exists \delta > 0$ such that

$$d'(x, y) < \delta \Rightarrow d(x, y) < \epsilon$$

and $\exists \delta' > 0$

$$d(x, y) < \delta' \Rightarrow d'(x, y) < \epsilon.$$

Continuity in Metric space Continuity of $f : V \mapsto W$ at v_0 can be defined through the following: $\forall \epsilon > 0$ there exists $\delta > 0$ such that

$$||v - v_0|| < \delta \Rightarrow ||f(v) - f(v_0)|| < \epsilon$$

this is implies by $||f(v)||_W \leq C||v||_V$. Also can be defined sequentially, i.e. if $x_n \to x$ and if $f(x_n) \to f(x)$ then f is continuous.

Holder's inequality: For all sequences we have the following:

$$\sum_{\mathbb{N}} |x_i x_j| \leq ||x||_p ||y||_q$$

if $x \in l^p$ and $y \in l^q$, and further $1/p + 1/q = 1$.

Young's Inequality

$$ab \leq a^p/p + b^1/q$$

or more generally:

$$ab \leq \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt$$

These are used to prove for example the Minkowski inequality for L^1, L^2 or the sequence space metrics etc.. See homework # 2.

2 Normed linear spaces = vector spaces

Definition We define a vector space V on a field F (usually on $\mathbb{R}$) with the following: First the properties of the vector space

1. $v + w = w + v$
2. $u + (u + w) = (u + v) + w$
3. $\exists 0 \in V, \; s.t. \; 0 + u = u$
4. $\exists - v \in V, \; s.t. \; -v + v = 0$

and operations involving the field elements $a, b \in F$:

1. $a(v + w) = aw + av$
2. $(a + b)v = av + bv$
3. $a(bv) = b(av)$
4. $1 \cdot v = v$ for 1 being the identity

The norm on this space satisfies the following conditions:

- $||v|| \geq 0$
- $||av|| = |a| \; ||v||$

- $||v+u|| \leq ||v|| + ||u||$

A metric is thus defined for $d(x,y) = ||x-y||$, so every vector space is a metric space.

Continuity of linear maps A linear map $f : (V, ||\cdot||_V) \mapsto (W, ||\cdot||_W)$ is defined with the following:

- $f(av+bw) = af(v) + bf(w)$
- $f(0_v) = 0_w$

The following are equivalent for normed spaces W, V:

- f is continuous
- f is continuous at 0.
- $\exists\ C < \infty$ such that $||f(v)||_W \leq C||v||_V$

The space of bounded linear maps is is denoted $\mathcal{L}(V, W)$ and the following is true:

- if $W, ||\cdot||_W$ is complete $\mathcal{L}(V, W)$ is complete
- We define the induced norm in this space as

$$||f||_{\mathcal{L}(V,W)} = \sup_{v \in V} \frac{||f(v)||_W}{||v||_V}$$

- $||AB||_{\mathcal{L}(V,W)} \leq ||A||_{\mathcal{L}(V,W)} ||B||_{\mathcal{L}(V,W)}$

E.g. For matrices we can fined through the above that $||A||_1$ is the max row sum, $||A||_2 = \sqrt{\lambda_{max}}$, where λ_{max} is the max eigenvalue of $A^T A$, and finally $||A||_\infty$ is the max col sum.

Dual space V^* This is the space is in one-to-one correpondence to the space of bounded linear maps $\mathcal{L}(V, \mathbb{R})$. I.e. $f \in \mathcal{L}(V, \mathbb{R}) \Leftrightarrow v \in V^*$.

Banach spaces are vector spaces that are complete, i.e. all Cauchy sequences converge in the space itself. **E.g.** $C[a,b]$ with $||\cdot||_\infty$ is complete as well as C_0 with $||\cdot||)\infty$. Counterexamples include $C[a,b]$ with L^2 norm or l_c (sequences with finite non-zero terms) with any l^p norm.

Semi-norms A semi-norm $p = L(v)$ for $v \in V$ and $L \in \mathcal{L}(V, \mathbb{R})$ satisfies

- $p(av) = |a|p(v)$
- $p(v+w) \leq p(v) + p(w)$

E.g. In particular we can build the l^1 norms on $\mathbb{R}^2$ through appropriate choices of p_1 and p_2 and take the supremum over seminorms. I.e. define $p_1 = x+y$ and $p_2 = x-y$ to produce that

$$||(x,y)||_1 = \sup_{i=1,2} p_i(v)$$

For the l^2 norm we need an arbitrary collection $p_a = x\cos a + y \sin a$. Its supremum over all $a \in [0, 2\pi]$ gives the 2-norm on vectors. In general we can build norms out of seminorms when $\forall\ v \neq 0 \Rightarrow p(v) > 0$. The mechanism we use is summing over many norms or taking a supremum.

3 Hilbert Spaces

Definition. Normed linear spaces that are complete. Norm must satisfy paralellogram law:

$$||x+y||^2 + ||x-y||^2 \leq 2||x||^2 + 2||y||^2$$

Cauchy-Schwartz says

$$| < x, y > | \leq ||x||_H ||y||_H$$

For M a closed linear space we can show that

$$M \oplus M^\perp = H$$

so we can decompose any $z \in H$ as $z = u + v$ for $u \in M$ and $v \in M^\perp$.

Also (for convex subspace as well), we can find unique $w \in M$ such that

$$\min_{v \in M} ||z - v|| = ||w - z||, \ w \in M$$

4 Topological Spaces

Definition The following 3 axioms define a topology $\mathcal{T}$ on a set X which is a collection of *open sets*,

- $\varnothing, X \in \mathcal{T}$
- Finite intersections are in space itself i.e. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$.
- Arbitrary unions are in space as well i.e. $U_\alpha \in T$ then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

An open U is defined such that $\forall y \in U$, $\exists V$ such that $y \in V \subseteq U$.

The **relative topology** is the topology that is restricted to a subset $YG \subseteq X$ such that

$$\mathcal{S} = \{H \subset Y \mid H = G \cap Y,\ G \in \mathcal{T}\}$$

A **Hausdorff** topology is such that for all $x, y \in X$ there exists open sets $U \ni x, V \ni y$ such that $U \cap V = \varnothing$.

A subset $V \subset X$ is a **neighborhood** of $x \in X$ if V is open and $V \ni x$.

Convergence in topological sense is defined through the following: $x_n \to x$ iff $\forall V \in \mathcal{N}(x)$ there exists N such that

$$n \geq N \Rightarrow x_n \in V$$

Continuity at a point is given by the following: for $f : X \mapsto Y$ then for f to be continuous at $x \in X$ then $\forall W \in \mathcal{N}(f(x)),\ \exists V \in \mathcal{N}(x)$ such that $f(V) \subseteq W$.

Continuity of functions in the topogical sense is implied by the following: Let $(X, \mathcal{T}$ and $(Y, \mathcal{S})$ are topological spaces, then $f : X \mapsto Y$ is continuous iff $f^{-1}(G) \in \mathcal{T}$ for all $G \in \mathcal{S}$. Note that sequential continuity does not directly imply the above (example is co-countable topology on $\mathbb{R}$).

A **Homeomorphism** is a function that is one-to-one and onto.

A **Neighborhood or local base** is defined through the following axioms:

- $V \in \mathcal{N}(x) \Rightarrow x \in V$
- If $U, V \in \mathcal{N}(x)$ then $\exists W \in \mathcal{N}(x)$ such that $W \subset V \cap U$.
- If $V \in \mathcal{N}(x)$ there exists $W \in \mathcal{N}(x)$ such that $W \subset V$
  - $W \subset V$
  - $y \in W \Rightarrow \exists U \in \mathcal{N}(y)$ and $U \subset V$.

We can define a topology using the local base i.e. O is open is $\forall x \in O$ there exists $V \in \mathcal{N}(x)$ such that $V \subseteq O$.

A **base** is defined through the following axioms for $B \in \mathcal{B}$:

- $\bigcup_{B \in \mathcal{B}} B = X$
- For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists $(B_3 \ni x) \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Additionally we can defined the unique topology $\mathcal{T}$ on X defined through the base above through

$$O \in \mathcal{T} \Rightarrow\ \forall y \in O, \exists B \in \mathcal{B} \text{ such that } (B \ni x) \subseteq O$$

In particular we can thus find that in $O \in \mathcal{T}$ then therefore $O = \cup_{U \in \mathcal{B}, U \subseteq O} U$.

The **weak topology** on X is defined as the topology that makes all linear functionals continuous, i.e. $f \in \mathcal{L}(V, \mathbb{R})$ is continuous.

Finally a countable local base makes the space **first countable**. A countable base or basis gives a **second countable**. In addition we prove that the space is second countable iff it is **separable** which implies that there is a countable set $\{x_n\} \subset M$ such that given any $x \in M$ and $\epsilon > 0$ then $\exists x_n$ such that $d(x_n, x) < \epsilon$.

5 Size of sets

5.1 Small/largeness

Dense set A set A is dense in X if $U \cap A \neq \emptyset$ for all $U^{open} \subset X$.

Interior points $A \subset X$, a point $x \in A$ is an interior point if $\exists U^{open} \ni x$ such that $U \subset A$. Therefore

$$A^o = \{x |\ x \in \text{ interior of} A\} = \bigcap_{V^{open} \subset A} V$$

Closure For $A \subset X$, $x \in \overline{A}$ if for all $V^{open} \ni x$, $V \cap A \neq \emptyset$ or

$$\overline{A} = \{x |\ x \in \text{ closure of} A\} = \bigcap_{A \subseteq F, F \text{closed}} F$$

It is a fact that $A^o \subset A \subset \overline{A}$, and so we can give the interpretation that A^o is open, $\overline{A}$ is closed.

Nowhere dense $A \subset X$ is nowhere dense if $\overline{A}$ has empty interior, i.e. $(\overline{A})^o = \emptyset$. Also, A is nowhere dense if $\overline{(A^c)^o} = X$ or its complement's interior is dense.

Meager or 1st category sets if A is the countable union of nowhere dense sets, then A is meager or 1st category. For example $\{r_n\}$ is an enumeration of the rationals and so since $\mathbb{Q}$ is a countable union of single element sets, then it is meager. A single element set is nowhere dense.

2nd category or residual sets Simply, the complement of a meager set is residual.

Density (again) If $A \subset X$ is dense in X then $\overline{A} = X$. Also, if $A \subset X$ is dense in B, then $B \subset \overline{A}$. Also, A is dense in X if and only if its complement has empty interior, i.e. $(A^c)^o = \emptyset$.

very small	nowhere dense
very large	dense interior
small	meager
large	residual
medium small	empty interior
medium large	dense

Measure zero A subset A of $\mathbb{R}^n$ has measure zero (or Lebesgue measure zero) if $\forall \epsilon > 0$, there is a countable collection $\{B_n\}$ of open balls $B_n = B(x_n, \delta_n) = \{x | \; ||x - x_n||_2 < \delta_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} B_n$$

and $\sum_{n=1}^{\infty}$ volume of $B_n \leq \epsilon$. Example: $\mathbb{Q}$ in $\mathbb{R}$. The cantor middle thirds set has measure zero even though this set has measure zero. Any countable union of elements have measure zero.

Generic sets Let $(X, \mathcal{T})$ be a topological space. A property P is generic (in the sense of **Baire** category) if

$$\{x | \; P(x) \text{ is true}\} \text{ is a dense residual set}$$

E.g. The property of being an irrational number is generic since the complement is meager ($\mathbb{Q}$) and $\mathbb{Q}^c$ is dense. The set of differentiable fcn's in $C[0,1]$ is not generic. The property of quadratic equations to have distinct roots is generic.

Zariski generic sets Let $X = \mathbb{R}^n$, and equip it with the Zariski topology. A set in this topology is considered open if its complement is the solution to a system of polynomial equations. The closed sets have measure zero. Therefore A property is generic if the set of x that have property P is a Zariski generic set. Equivalently, the set not having the property is given by the solution set of finitely many algebraic equations.

E.g. A quadratic equation $x^2 + bx + c$ has non-distinct roots when $b^2 - 4c = 0$, therefore it is Zariski closed, and thus the property of *not* having this property is Zariski generic.

Baire category theorem?

5.2 Measure theory

A **Measurable space** is a set X together with a collection of subsets $\mathcal{B}$ of X satisfying:

$$\begin{aligned} & X \in \mathcal{B}, \emptyset \in \mathcal{B} \\ & \text{if } A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B} \\ & \text{if } A_1, A_2 \dots \in \mathcal{B} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{B} \end{aligned}$$

$\mathcal{B}$ is called a σ-algebra. Also closed under countable intersections.

A **Measure** μ satisfies the following relations on a measurable space $(X, \mathcal{B})$,

$$\begin{aligned} & \mu : \mathcal{B} \mapsto [0, \infty] \\ & \text{such that } \mu(\emptyset) = 0 \\ & \text{and } \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \end{aligned}$$

whenever the set $\{A_n\}$ are mutually disjoint and belong to $\mathcal{B}$.

E.g. we can prove the following: Let $A \subset B$ then $B = (B - A) \cup A$ and $\mu(A) = \mu(B - A) + \mu(A)$. Also if $\mu(A) \leq \mu(B)$. More importantly, if $A_n \in \mathcal{B}$ and $A_1 \supset A_2 \dots$ then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

also let $B_k \in \mathcal{B}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n)$$

The Borel σ-algebra is the smallest such σ-algebra containing all the open sets in a topology $\mathcal{T}$.

disjointness is important in proving the results concerning the measure. Therefore if given a general set $\{E_i n\} \in \mathcal{B}$, then define

$$F_1 = E_1, \; F_2 = E_2 \cap F_1^C, \text{or } F_j = E_j \cap \bigcap_{n=1}^{j-1} F_n$$

6 Convergence and compactness

6.1 Convergence and completeness

Convergence in topological space for a sequence $x_n \to x$ is given by for each open sets or neighborhoods of x, V, then $\exists N$ such that

$$n > N \Rightarrow x_n \in V$$

Convergence in metric space (X, d) assumes more structure, i.e. for all $\epsilon > 0, \exists N_\epsilon$ such that

$$n > N_\epsilon \Rightarrow d(x_n, x) < \epsilon$$

Cauchy convergence A sequence is Cauchy if $\forall \epsilon > 0, \exists N_\epsilon$ such that

$$m, n \geq N_\epsilon \Rightarrow d(x_n, x_m) < \epsilon$$

A metric space is called **complete**, if every Cauchy sequence converges in the cited metric space. E.g. the rationals are incomplete, consider $x = 1$, $x_n = x_{n-1} + 1/x_n$, this sequence converges to $\sqrt{2} \notin \mathbb{Q}$. In function spaces the sequence $t^n \in C[0,1]$ is Cauchy in metric L^1 but it converges to a discontinuous function.

Conversely, $\mathbb{R}^2$ with euclidean norm is complete or even $l^p(\mathbb{R}, \mathbb{Q})$. Also, $C[0,1]$ under the supremum or L^∞ norm.

Contraction mapping theorem For a complete metric space (M, d), let $f : M \mapsto M$ be a continous function and suppose that $\exists 0 < k < 1$ such that

$$d(f(x), f(z)) \leq kd(x, z), \forall x, z \in M$$

then there exists a unique $y \in M$ such that $f(y) = y$.

For example Newton's method requires the computation $x_{n+1} = x_n - g(x_n)/g'(x_n)$, so we can show that $f(x) = x - g(x)/g'(x)$ is a contraction mapping for $g \in C^2[0,1]$ (2× differentiable).

Baire category theorem In a complete metric space, every residual set is dense.

Further, this implies that every complete metric space is *not* a countable union of disjoint nowhere dense sets.

6.2 Completions

An **equivalence relation** on X is a subset of $X \times X$, i.e. a subset of pairs $\{(x_1, x_2) | x_1, x_2 \in X\}$ or written $x_1 \sim x_2$ that satisfy the conditions:

$$(i)\ x_1 \sim x_1$$
$$(ii)\ x_1 \sim x_2 \Rightarrow x_2 \sim x_1$$
$$(iii)\ x_1 \sim x_2, x_2 \sim x_3 \Rightarrow x_1 \sim x_3$$

An **equivalence class** of x denoted $[x]$, is the set of points equivalent to it, i.e.

$$[x] = \{y |\ y \sim x\}$$

The **completion** of a possible incomplete metric space (M, d) is defined in the following way: if $\{y_n\}$ and $\{x_n\}$ are two Cauchy sequences in M we say define the equivalence relation between sequences as $\{x_n\} \sim \{y_n\}$ if $d(x_n, y_n) \to 0$ as $n \to \infty$. $\overline{M}$ is the collection of all the equivalence classes of Cauchy sequences in M. For such a space define the distance metric as $\overline{d}$,

$$\overline{d}([\{x_n\}], [\{z_n\}]) = \lim_{n \to \infty} d(x_n, z_n)$$

Misc. things we can prove: $\overline{d}$ always exists, the specific Cauchy sequence does not affect the value of $\overline{d}$, $\overline{d}$ is a metric, and $(\overline{M}, \overline{d})$ is a complete metric space. Defining the constant sequence x, then the $\overline{d}$ metric collapses to the metric in d. Also, M is dense in $\overline{M}$.

6.3 Compactness

The property of **compactness** *morally* approximates the properties of finite sets, in fact all finite element sets are compact. More generally we define a compact set on metric spaces according to the following definition: For a compact set $A \subset X$, any sequence $\{x_n\} \in A$ has a *convergent subsequence* $\{x_{n(k)}\}$

E.g. trivially, $\{1, -1, 1, -1 \ldots\}$ has a convergent subsequence to ± 1. The real numbers are compact, consider $x_n = n$, it has no convergent subsequence.

Things we can prove: a compact metric space is complete. A compact subset A of a metric space must be closed and bounded. The converse is not true. $\mathbb{R}$ is closed unto itself but clearly not compact. Also, you can be bounded but not compact, i.e. $(0, 1]$.

A **bounded** set A is defined by the parallel constructions that

$$\exists K > 0, \text{ such that } d(x, y) < K,\ \forall x, y \in A$$
$$\exists x_0 \in A, P < \infty \text{ such that } d(x_0, y) < P, \forall y \in A$$

Heine-Borel Theorem A subset A of $\mathbb{R}^n$ is compact iff it is closed and bounded. A corollary states that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Now we will deal with properties of **continuous functions on compact sets**, i.e.

- For $f : M \mapsto \mathbb{R}$ continuous, and so $\exists x_\pm$ such that for all $x \in M$ then

$$a = f(x_-) \leq f(x) \leq f_+(x) = A$$

There is also a parallel definition of compactness for a topological space $(X, \mathcal{T})$, i.e. consider $\{U_\alpha\}$ a possibly uncountable open covering of a set $A \subset X$, i.e.

$$A \subset \bigcup_\alpha U_\alpha$$

then if A is compact there exists a *finite* subcovering, i.e.

$$A \subset \bigcup_{n=1}^{N} U_{\alpha_n}$$

for all open coverings. The definition for a space itself is directly analogous.

A related property is the so-called **finite intersection** property, i.e. if $\{F_\alpha\}$ is a collection of closed subsets of a topological space $(X, \mathcal{T})$ then the collection has this property if a finite intersection of such sets is non-empty:

$$\bigcap_{i=1}^{N} F_{\alpha_i} \neq \emptyset$$

If a topological space is compact then for any collection of closed sets that has the finite intersection property, then

$$\bigcap_\alpha F_\alpha \neq \emptyset$$

Related facts: A closed subset of a compact topological space is compact. ALso, a compact subset of a Hausdorff space is closed.

6.4 Uniformity

Definition of **uniform continuity**: Let $f : (M, d) \mapsto (M', d')$ (metric spaces) then f is uniformly continuous if $\forall \epsilon > 0,\ \exists \delta > 0$, such that

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon$$

Note δ depends only on ϵ, and not x which refers to pointwise convergence.

Fact: A continuous function from a compact metric space inherits some structure from the domain, and thus f is uniformly continuous.

Now we define **pointwise convergence** of functions: Let (M, d) and (M', d') be metric spaces, then $\{f_n\}$ are a sequence of functions from M to M' and suppose that $f : M \mapsto M'$, then if

$$\lim_{n\to\infty} f_n(x) = f(x) \forall x \in M$$

then f_n approaches f **pointwise**!

Now for **uniform convergence**. Then if $\forall \epsilon > 0, \exists N$ such that

$$n > N \Rightarrow d'(f_n(x), f(x)) < \epsilon, x \in M$$

The **Weierstrass M-test**: Suppose f_n are a sequence of functions in (M, d) and suppose that

$$||f_n||_\infty \leq a_n, \text{with } ||a_n||_1 < \infty$$

then the partial sums converge uniformly for large enough m, i.e.

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly}$$

The **uniform limit of continuous functions is continuous**! It means that $C(M)$ is complete for (M, d) a metric space.

Term-by-term differentiation Suppose $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on some interval $J \subset \mathbb{R}$ and also the sum of the derivatives, $g(x) = \sum_{n=1}^{\infty} f'_n(x)$ converges uniformly as well, then $f'(x) = g(x)$.

This is related to the fact that the property of *not* having a derivative is generic.

Intergration of a uniformly convergent sequence is allowed but can be relaxed (using Lebesgue integration). At present, knowing that $\{f_n\}$ is a uniformly convergent sequence on $[a, b]$ then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Now we turn to parametrized integrals. So we define uniform convergence for integrals, i.e.

$$\int_{\mathbb{R}} f(t, x)dx = F(t)$$

converges uniformly to $F(t)$ for $t \in [a, b] \subseteq \mathbb{R}$, if for every $\epsilon > 0$ there is an R_0 depending on ϵ but not t, such that for all $R > R_0$, $\forall t \in [a, b]$,

$$\left| \int_{-R}^{R} f(t, x)dx - F(t) \right| < \epsilon$$

Differentiating parametrized integrals, can be done under the hypothesis that

$$\int_{\mathbb{R}} f(t,x)dx = F(t), \int_{\mathbb{R}} \frac{\partial f}{\partial t} dx = G(t) \text{ converge uniformly}$$

then $F'(t) = G(t)$.

7 Lebesgue Integration

The motivation for developing the idea of Lebesgue integration comes from consideration of the completion of $C[0,1]$ under the L_1 metric for example. It turns out these are measurable functions and the metric is Lebesgue integration (?).

Define $U(f,\Delta) = \sum_{i=0}^{n} \max_{x\in(t_i,t_{i+1})} f(t)(t_{i+1} - t_i)$ and $L(f,\Delta) = \sum_{i=0}^{n} \min_{x\in(t_i,t_{i+1})} f(t)(t_{i+1} - t_i)$. These are approximations to the integral/area under the curve of f. Therefore the limiting process of the can be defined:

$$U(f) = \inf_{\Delta} U(f,\Delta),\ L(f) = \sup_{\Delta} L(f,\Delta)$$

In particular if f is Riemann integrable then $U(f) = L(f)$ and so for any continuous function this condition is met since they will be Riemann integrable. Can also that a general function on $[0,1]$ is Riemann integrable if and only if the set of points of discontinuity of f is of Lebesgue measure zero.

We define **measurable functions**, take $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be measurable spaces, then a function $f : X \mapsto Y$ is measurable if $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{C}$.

This definition can be extended to the case that really interest us, which means that $Y = \mathbb{R}$. Therefore first define the extended real numbers:

$$\mathbb{R}_{ex} = \mathbb{R} \cup -\infty, \infty = [-\infty, \infty]$$

with the following rules : $a \in real$, $a \pm \infty = \pm\infty$, if $a > 0$, then $a \times \pm\infty = \pm\infty$, $0 \times \pm\infty = 0$, $\infty - \infty$ is undefined.

So consider $(X,\mathcal{B})$ a measurable space. Then $f : X \mapsto \mathbb{R}_{ex}$ is measurable if the set

$$\{t|\ f(t) < \alpha\} \in \mathcal{B},\ \alpha \in \mathbb{R}$$

This property is inherited to a sequence of such functions, i.e. if $\{f_n\}$ are measurable then if $f_n \to f$ pointwise, then f is measurable.

7.1 Lebesgue integral

From here on we expect $(X,\mathcal{B},\mu)$ is the domain. Firstly lets define the characteristic function, i.e. $\chi_A(x)$ such that

$$\chi_A(x) = \begin{cases} 1,\ x \in A \\ 0,\ x \notin A \end{cases}$$

A **nonnegative simple function** Let $A_1 \dots A_n \subset X$ be pairwise disjoint measurable sets, a function

$$\phi(x) = \sum_{j=1}^{n} \alpha_j \chi_{A_k}$$

then ϕ is a simple function (step function). Within the Lebesgue theory we expect that

$$\int \phi = \sum_{j=1}^{n} \alpha_j \mu(A_j)$$

The idea is to approximate the integral of a general function f with successive simple functions of increasing n.

So let $f : X \mapsto [0,\infty]$ and measurable, then its Lebesgue integral is given by

$$\int f = \sup_{0\le\phi\le f, \phi \text{ simple}} \int \phi$$

Now consider a general function that can take negative values, then we can integrate each part separately, i.e. $f = f_+ - f_-$. Work out which one is positive/negative! hah!

There is a analogy to sequence where the measure is the number of points in a set. The function is the sequence. See page 355.

For non-negative fcn's the Lebesgue integral retain linearity and positive definiteness, and the the integral is zero iff f is zero almost everywhere = the set where the function is not zero is measure zero.

7.2 Convergence theorems

Fatou's Lemma We introduce first the idea of **limit inferior** and **limit superior**. In the sense of real numbers

$$\liminf a_n = \text{least limit point of the sequence}$$

or the least point such that for any open set containing it, there are only finite number of points to the left. Similarly for $\limsup a_n$,

$$\limsup a_n = \text{greatest limit point of the sequence}$$

If the series converges, then these two things are equal. For **Fatou's lemma** we use the notion that things can't escape to ∞, i.e,

$$\int_X \lim inf f_n d\mu \leq \lim\inf \int f_n d\mu$$

Monotone convergence theorem states that if we have a sequence of non-negative measurable functions $\{f_n\}$ such that $f_{n+1}(x) \geq f_n(x)$ for each x (a non-decreasing set) and $\lim f_n = f$ (pointwise) then

$$\lim \int f_n dx = \int \lim f_n dx$$

Note: power series can be integrated term by term in compact subsets of their interval of convergence.

Dominated convergence theorem states that for a sequence $\{f_n\}$ that can be dominated $|f_n| \leq g$ and $\lim f_n = f$ a.e. and g is integrable, then

$$\lim \int f_n dx = \int \lim f_n dx$$

7.3 Lebesgue spaces

Let $(X, \mathcal{B}, \mu)$ be a measure space. $L^P(x)$ $(1 \leq p < \infty)$ is the linear space of equivalence classes of measurable, L^p-integrable functions from X to $\mathbb{R}_{ex}$. Two functions are measurable if they differ only one a set of measure zero. Therefore $0 \sim \{1, x \in \mathbb{Q}, 0, x \in \mathbb{Q}^c\}$.

The case of $p = \infty$ is different. We define that $L^\infty(X, d\mu)$ is the set of equivalence classes of measurable functions with the following property:

$$\exists M > 0, \text{ such that } |f(x)| \leq M, \text{for a.e.} x \in X$$

The equivalence class is as before, $f \sim g$ if they differ only on a set of measure zero. The norm is actually called the essential supremum, i.e.

$$\operatorname{ess\,sup} f = \inf M$$

Cauchy sequence in L^∞: Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(X)$, then there is a set $E \subset X$ of measure zero with the property that $\forall \epsilon > 0$, $\exists N$ such that

$$|f_n(x) - f_m(x)| > \epsilon, \forall x \in X - E \text{ whenever } n, m > N$$

Another important aspect is the **Riesz-Fischer Theorem**, for $1 \leq p \leq \infty$, $L^p(\infty)$ is a complete metric space. Also, $L^p(\infty)$ are normed linear spaces.

Chebyshev's Inequality Let f be a measurable, extended real-valued function on a measure space $(X, \mathcal{B}, \mu)$. For every $\epsilon > 0$,

$$\mu(\{x|\ |f(x)| > \epsilon\}) \leq \frac{1}{\epsilon} \int |f| d\mu$$

Regular Let $(X, \mathcal{T})$ be a topological space and suppose that X is locally compact: every $x \in X$ has a neighborhood whose closure is compact. Let μ be a measure defined on $\mathcal{B}$ (Borel σ-algebra). Then μ is **regular** if for every measurable set E,

$$\begin{aligned}\mu(E) &= \inf\{\mu(O)|\ O \supseteq E,\ O^{open}\} \\ &= \sup\{\mu(K) \mid K \subseteq E, K \text{compact}\}\end{aligned}$$

It means that the measure of a set E can be approximated by a compact set enclosing it, or open sets contained in E. An alternate definition (or consequence of the above): A measure is Borel regular if $\forall E$ measurable and $\epsilon > 0$,

$$\exists O^{open}, O \subseteq E, K^{compact} \supseteq B \text{ such that } \mu(O-E) < \epsilon,\ \mu(B-K) <$$

The crowning achievement: **denseness** of C_0^∞, if μ is Borel regular measure on X then continuous functions with compact support are dense in $L^p(X)$ for $1 \leq p < \infty$. Shankar's advice:

- Convergence in L^p (or more generally, convergence of integrals) does not imply pointwise convergence. However, one can usually extract subsequences that converge pointwise.

  E.g. The halves example. They converge in L^p but do not converge pointwise. However, we can extract a subsequence that converges. Since it's a Cauchy sequence, the sequence converges to this.

- Chebyshev's inequality. See above.

- L^1 (or more generally L^p) is a complete normed linear space. For $p < \infty$, it is the completion of the $C_0^\infty(R)$ with respect to the L^p metric. In particular, this implies that the compactly supported continuous (and also smooth) functions are dense in L^p.

References

Principles of Analysis, H. Flaschka, `University of Arizona`