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AN OPERATOR EQUATION AND BOUNDED SOLUTIONS OF INTEGRO-DIFFERENTIAL SYSTEMS*

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Abstract. The main result gives conditions under which (locally) a one-to-one, bicontinuous correspondence exists between bounded solutions (or bounded solutions tending to zero as $t \to +\infty$) of a linear, integro-differential system of Volterra type and such solutions of perturbations of the system. The perturbations are allowed to be of any functional type which satisfy a local Lipschitz condition near the origin. Certain recently proved stability results for such systems are special cases. The results also constitute a generalization of similar results for ordinary differential equations, which motivate the approach and proofs. The proofs rely on an abstract lemma proved for a certain operator equation. In order to apply the perturbation theorems some results are also given concerning bounded solutions of linear integro-differential systems. An application is made to Volterra's predator-prey population dynamics model with hereditary effects where it is shown, for certain specific, but reasonable hereditary kernels, that the critical (or saturation point) of the system is unstable.

Introduction. Our main purpose is to investigate the existence of bounded solutions of the $n \times n$ system of integro-differential equations

$$\text{(P)} \qquad x'(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)\,ds + h(t)(x) + g(t), \qquad t \geqq t_0,$$

where $h(t)(0) \equiv 0$, $t \geqq t_0$, which is to be considered a perturbation of the linear homogeneous system

$$\text{(H)} \qquad y'(t) = A(t)y(t) + \int_{t_0}^{t} B(t,s)y(s)\,ds, \qquad t \geqq t_0.$$

The goal is to place conditions on h and on (H), or more precisely on the related nonhomogeneous system

$$\text{(NH)} \qquad z'(t) = A(t)z(t) + \int_{t_0}^{t} B(t,s)z(s)\,ds + g(t), \qquad t \geqq t_0,$$

under which it is possible to assert that (locally) there is a one-to-one correspondence between the bounded solutions of (P) and those of (H). The perturbation term h is to be thought of as an operator which maps the set of functions defined for $t \geqq t_0$ into itself and which is in some sense small; this will be made precise below. Typical perturbations are of the form

$$h(t)(x) = h(t, x(t)) \quad \text{or} \quad \int_{t_0}^{t} K(t,s,x(s))\,ds \quad \text{or} \quad x(t)\int_{t_0}^{t} K(t,s,x(s))\,ds.$$

Obviously perturbations of Fredholm type could also be considered. Specific conditions will be placed on the $n \times n$ matrices A and B below.

First, in § 1 we will state and prove a result for an abstract operator equation which, when applied to (P) in § 2 will lead to our main results. In § 3 we will study the linear nonhomogeneous system (H) with regard to the hypotheses for our

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main results. Finally in § 4 the important special case $A(t) \equiv A =$ const. and $B(t, s) \equiv B(t - s)$, which is the form of the vast majority of applications, will be considered. Included in § 4 is an application to Volterra's equations for predator-prey population dynamics with hereditary effects where, amongst other things, it is shown that under certain reasonable assumptions on the hereditary factor such populations will not have a stable critical or saturation point.

Our approach and results for (P) are motivated by (and generalize) certain results for differential equations ($B \equiv 0$). See, for example, [3], [8]. They also serve to generalize certain stability results for integro-differential equations [6], [7], [9], [11]. Our Lemma 1 below bears an interesting relationship to the idea of admissibility of linear operators and the results of Corduneanu and Miller [4], [10], [12]. Lemma 1 is independent of the abstract results of Miller in [10] and the admissibility approach in general in that it deals with closed operators on spaces which are not necessarily complete as opposed to continuous linear operators on Banach spaces.

1. An operator equation. Consider the equation

$$Lx = f(x), \tag{1.1}$$

where L is a linear operator with domain $D(L)$ and range $R(L)$ contained in Banach spaces X and Y respectively and where $f(x)$ is an operator from X into Y. Let $N(L)$ denote the null space of L. Our goal is to obtain a correspondence between solutions of (1.1) and $N(L)$ by making suitable assumptions on L and f. First, we assume the following.

H1. L is closed on $D(L)$ and there exists a subspace $S \subseteq D(L)$ such that the restriction of L to S (denoted L_s) is closed and one-to-one and has closed range.

Here the domain $D(L)$ and the subspace S are purposively not assumed to be complete as this will be the case in our applications to (P) below. Let $R(L_s)$ be the range of L_s and set $\Sigma(r) = \{x \in X : |x|_X \leqq r\}$. Concerning the operator f we assume, without loss of generality, that $f(x) = h(x) + g$ where $h(0) = 0$ and $g \in Y$; in addition we assume the following hypothesis.

H2. h maps $D(L)$ into $R(L_s)$ continuously in such a way that for some constants θ and r, $0 \leqq \theta < +\infty$, $0 < r \leqq +\infty$, we have $|h(x) - h(y)|_Y \leqq \theta|x - y|_X$ for all $x, y \in D(L) \cap \Sigma(r)$.

Under H1 it follows by the closed graph theorem that L_s has a bounded inverse L_s^{-1}. From this we can conclude the following basic lemma.

LEMMA 1. *Suppose* H1 *and* H2 *hold. Suppose also that* θ *in* H2 *satisfies* $\theta|L_s^{-1}| < 1$. *Then there exists a constant* $c > 0$ *such that for each* $g \in R(L_s)$ *satisfying* $|g|_Y \leqq cr$, *a one-to-one bicontinuous mapping* Q *exists from the set* $N(L) \cap \Sigma(cr)$ *into the set of solutions of* (1.1) *contained in* $D(L) \cap \Sigma(r)$.

Proof. We first show that Q is well-defined. Given $n \in N(L) \cap \Sigma(cr)$, define the operator $T: D(L) \to D(L)$ by $Tx \equiv n + L_s^{-1}f(x)$. For $x \in \Sigma(r)$ it follows from H2 (with $y = 0$) that

$$|Tx|_X \leqq [c(1 + |L_s^{-1}|) + |L_s^{-1}|\theta]r.$$

Thus, if we choose $c < (1 - |L_s^{-1}|\theta)(1 + |L_s^{-1}|)^{-1}$, then T maps $D(L) \cap \Sigma(r)$

into itself. Moreover, by H2 we have $|Tx - Ty|_Y \leqq |L_s^{-1}|\theta|x - y|_X$ and hence T is a contraction on $D(L) \cap \Sigma(r)$. Choosing $x_1 \in D(L) \cap \Sigma(r)$ and setting

$$x_n = Tx_{n-1} \in D(L) \cap \Sigma(r), \quad n \geqq 2,$$

we know that $x_n \to x_0$ in X and that $f(x_n) \to f(x_0) \in R(L_s)$ in Y (by H2). Since $Lx_n = LTx_{n-1} = LL_s^{-1}f(x_{n-1}) = f(x_{n-1}) \to f(x_0)$ in Y and since L is closed on $D(L)$, it follows that the point x_0 lies in $D(L) \cap \Sigma(r)$ and $Lx_0 = f(x_0)$; that is, x_0 solves (1.1) and we have a well-defined function $Q : Q(n) = x_0$.

Suppose $n_i \in N(L) \cap \Sigma(cr)$ for $i = 1, 2$ and $Q(n_1) = Q(n_2) = x$; i.e., $x = n_i + L_s^{-1}f(x)$ for $i = 1, 2$. By subtraction we find that $n_1 = n_2$ and that Q is consequently one-to-one. Finally, Q and Q^{-1} are continuous as is shown by the following inequalities:

$$\begin{aligned}|Q(n_1) - Q(n_2)|_X = |x_1 - x_2|_X = |Tx_1 - Tx_2|_X &\leqq |n_1 - n_2|_X + |L_s^{-1}|\theta|x_1 - x_2|_X \\ &= |n_1 - n_2|_X + |L_s^{-1}|\theta|Q(n_1) - Q(n_2)|_X\end{aligned}$$

or

$$|Q(n_1) - Q(n_2)|_X \leqq (1 - |L_s^{-1}|\theta)^{-1}|n_1 - n_2|_X;$$

and

$$\begin{aligned}|Q^{-1}(x_1) - Q^{-1}(x_2)|_X &= |n_1 - n_2|_X = |x_1 - Tx_1 - x_2 + Tx_2|_X \\ &\leqq |x_1 - x_2|_X + |L_s^{-1}|\theta|x_1 - x_2|_X \\ &= (1 + |L_s^{-1}|\theta)|x_1 - x_2|_X. \qquad \square\end{aligned}$$

Remarks. (a) If $R(L)$ is closed in Y and $N(L)$ admits a projection, then we may take $S = M$ in H1 and Lemma 1 where $D(L) = N(L) \oplus M(L)$. Then L_s^{-1} is the pseudo-inverse of L on $D(L)$ and, by the closed graph theorem, L is continuous on $D(L)$. These circumstances do not hold, however, in our application to (P) below.

(b) We can further assert that if $|g|_Y \leqq cr/|L_s^{-1}|$, then there exists a constant $r^* > 0$ such that the range of Q contains *all* solutions of (1.1) contained in $D(L) \cap \Sigma(r^*)$. To see this, choose r^* so small that $|x - L_s^{-1}f(x)|_X \leqq cr$ for $|x|_X \leqq r^*$. This is possible by the way g is chosen since $I - L_s^{-1}f$ is continuous. Let $x \in D(L) \cap \Sigma(r^*)$ be a solution of (1.1) and define $n = s - L_s^{-1}f(x)$ which, by the assumption made, lies in $N(L) \cap \Sigma(cr)$. Thus, there exists a unique solution $x' \in D(L) \cap \Sigma(r)$ of (1.1) such that $x' = Q(n)$; i.e., $x' = n + L_s^{-1}f(x')$. But then $x' - x = L_s^{-1}(f(x') - f(x))$ which implies $|x' - x|_X \leqq |L_s^{-1}|\theta|x' - x|_X$ or $x' = x$ in as much as $|L_s^{-1}|\theta < 1$. Hence, x is in the range of Q.

(c) If it is assumed that S is a Banach space and that $R(L)$ is contained in a Banach subspace Y^* of Y, then Theorem 1 can be proved with X and Y taken as Fréchet spaces (instead of Banach spaces) whose respective topologies induce topologies on S and Y^* weaker than their respective norm topologies. In this case L is continuous by the closed graph theorem and one obtains from this modification of Theorem 1 an alternate statement of a theorem of Miller [10, Thm. 1]. Again, however, in our applications S is not complete.

2. Integro-differential equations. We return now to systems (P), (H) and (NH) where we assume the following.

H3. $g(t)$ and $A(t)$ are locally integrable in $t \geqq t_0$ and $B(t, s)$ is locally in L^1 in (t, s), $t \geqq s \geqq t_0$.

Under these conditions, the Volterra integral equation obtained from (NH) by integration has a kernel $k(t, s) = A(s) + \int_s^t B(r, s)\, dr$ for which, it is not difficult to see, conditions sufficient for the existence and uniqueness of a continuous solution for $t \geqq t_0$ (as given by Miller in [12]) are fulfilled for each initial vector $x(t_0) = x_0 \in R^n$ and each $g(t)$. This continuous solution, by virtue of the fact that it solves this integral equation and that $k(t, s)$ has the properties described in H3, is in fact absolutely continuous and consequently is the unique solution of (NH) for $t \geqq t_0$ and $x(t_0) = x_0$. Our goal now is to apply Lemma 1 to the perturbed system (P).

Let

$$BC = \{x(t) \in C[t_0, +\infty) : |x|_0 = \sup_{t \geqq t_0} |x(t)| < +\infty\}$$

and L^p, $1 \leqq p < +\infty$, be the Banach space of functions defined and measurable for $t \geqq t_0$ for which $|x|_p = \int_{t_0}^{+\infty} |x|^p\, dx < +\infty$. For convenience, we let L^∞ also denote BC and $|x|_\infty = |x|_0$. We take $X = BC$ and $Y = L^p$, $1 \leqq p \leqq +\infty$, in Lemma 1. Define the linear operator L by

$$Lx \equiv x' - A(t)x - \int_{t_0}^t B(t, s)x(s)\, ds$$

whose domain we take to be the linear subspace $D^p(L) = \{x \in BC : x(t) \text{ is absolutely continuous for } t \geqq t_0 \text{ and } Lx \in L^p\}$. (By a solution of (P), (H), or (N) we mean an absolutely continuous function satisfying the corresponding system for almost all $t \geqq t_0$.) Define X_1 to be those vectors in R^n which, as initial conditions at t_0, give rise to bounded solutions of (H); X_1 is clearly a linear subspace of R^n. Let X_2 be any space supplementary to X_1: $R^n = X_1 \oplus X_2$; and let P_i be the projection of R^n onto X_i. In H1 we take S to be $S^p = \{x \in D^p(L) : x(t_0) \in X_2\}$ which is easily seen to be a subspace of $D^p(L)$. In order to fulfill H1 we assume the following.

H4^p. for each $g(t) \in L^p$, $1 \leqq p \leqq +\infty$, there exists at least one bounded solution $z \in BC$ of (NH).

Under this hypothesis the range R^p of L restricted to S^p is all of $Y = L^p$ and hence is closed. For by H4^p, given $g \in Y$ there exists $z \in D^p(L)$ such that $Lx = g$ and if $y(t) \in BC$ is the unique solution of (H) satisfying $y(t_0) = P_1 z(t_0)$, then $x = z - y \in S^p$ and $Lx = g$. Moreover, L is one-to-one on the subspace S^p for if $Lx_1 = Lx_2$ for $x_1, x_2 \in S^p$, then $L(x_1 - x_2) = 0$ and $x_1 - x_2 \in S^p$, which means $y = x_1 - x_2$ is a bounded solution of (H) with initial state in X_2. Since X_2 is supplementary to X_1 it must be that $y = 0$.

Finally all that remains in order to show that H1 is fulfilled is that L is closed on $D^p(L)$ and S^p. To this end suppose $x_n \in D^p(L)$ and $g_n = Lx_n$ converge in BC and L^p respectively to $x^0 \in BC$ and $g^0 \in L^p$. Integrating $g_n = Lx_n$, we have

$$x_n(t) = x_n(t_0) + \int_{t_0}^t \left[A(s) + \int_s^t B(r, s)\, dr \right] x_n(s)\, ds + \int_{t_0}^t g_n(s)\, ds.$$

For fixed, but arbitrary $t \geqq t_0$, we find (using H3 and the dominated convergence theorem) that

$$x^0(t) = x^0(t_0) + \int_{t_0}^{t} \left[A(s) + \int_{s}^{t} B(r, s)\, dr \right] x^0(s)\, ds + \int_{t_0}^{t} g^0(s)\, ds$$

and consequently $x^0(t)$ is absolutely continuous and solves $Lx^0 = g^0$. This proves that L is closed on $D^p(L)$. If on the other hand $x_n(t) \in S^p$, then in addition to $x^0(t) \in D^p(L)$, it is obvious that $x_n(t_0) \in X_2$ implies that $x^0(t_0) \in X_2$ and hence $x^0(t) \in S^p$; i.e., L is also closed on S^p.

Having fulfilled H1, we can assert the conclusion of Lemma 1 for (P) provided the perturbation term h and the nonhomogeneous term $g(t)$ satisfy the necessary conditions. The following two theorems contain our main results.

THEOREM 1. *Suppose* H3 *and* H4^p *hold. Further suppose $h(t)(x)$ maps BC into L^p, $1 \leqq p \leqq +\infty$, in such a way that for some constant r, $0 < r \leqq +\infty$,*

$$|h(t)(x) - h(t)(y)|_p \leqq \theta |x - y|_0 \tag{2.1}$$

holds for all $x, y \in BC$ satisfying $|x|_0, |y|_0 \leqq r$. Then there exist three positive constants a, b and θ^0 with the following properties: if $\theta \leqq \theta^0$ then

(i) *for every $g \in L^p$, $|g|_p \leqq a$, there exists a one-to-one bicontinuous correspondence Q between the bounded solutions $y \in BC$ of* (H) *satisfying $|y(t_0)| \leqq b$ and the bounded solutions $x \in BC$ of* (P) *satisfying $|x|_0 \leqq r$, $|P_1 x(t_0)| \leqq b$; and*

(ii) *the correspondence Q is such that if $x = Qy$, then $P_1 x(t_0) = y(t_0)$.*

Proof. For all the bounded solutions $y(t)$ of the linear homogeneous system (H) it is possible to assert that $|y|_0 \leqq M|y(t_0)|$ for some constant $M > 0$. The stated assumption (2.1) on h allows us to apply Lemma 1 in the context described above. Let c be the constant whose existence is guaranteed by Lemma 1 and take $\theta^0 = \frac{1}{2}|L_s^{-1}|^{-1}$ (where $S = S^p$), $a = cr$, and $b = M^{-1}cr$. Given a bounded solution $y(t)$ of (H), $|y(t_0)| \leqq b$ (hence, $y(t_0) \in X_1$), it follows that $|y|_0 \leqq cr$ and by Lemma 1 there exists a unique corresponding solution $x = Qz$ of (P) satisfying $|x|_0 \leqq r$; moreover, Q is invertible and bicontinuous. Referring to the proof of Lemma 1, $x = y + L_s^{-1} f(t)(x)$ and, hence, $P_1 x(t_0) = P_1 y(t_0) = y(t_0)$ since y being bounded implies $y(t_0) \in X_1$ and since $L_s^{-1} f(t)(x) \in S^p$ implies that $L_s^{-1} f(t)(x)$ at t_0 lies in X_2. Finally, Q is onto the set of solutions of (P) as described in the theorem, for if x is such a solution of (P) ($|x|_0 \leqq r$ and $|P_1 x(t_0)| \leqq b$) then we may define $y = x - L_s^{-1} f(t)(x)$ and find that y is a bounded solution of (H) satisfying $|y(t_0)| = |P_1 y(t_0)| = |P_1 x(t_0)| \leqq b$. Hence, $|y|_0 \leqq cr$ and $x' = Qy$ exists. But then $x' = y + L_s^{-1} f(t)(x')$ and hence $x - x' = L_s^{-1}[f(t)(x) - f(t)(x')]$ and $|x - x'|_0 \leqq |L_s^{-1}|\theta^0 |x - x'|_0$ by (2.1). Since $|L_s^{-1}|\theta^0 = \frac{1}{2}$, we conclude $x = x'$ and that Q is onto. □

As a second application of Lemma 1 we consider the question of the existence of bounded solutions of (P) which in addition tend to zero as $t \to +\infty$. We define $BC_0 = \{x \in BC : |x(t)| \to 0 \text{ as } t \to +\infty\}$ and let $X_1^0 \subseteq R^n$ be the linear space of initial vectors at $t = t_0$ which give rise to solutions of (H) in BC_0. Let $X_2^0 \subseteq R^n$ be such that $X_1^0 \oplus X_2^0 = R^n$ and P_i^0 be the projections of R^n onto X_i^0. If we take $X = BC_0$ and $Y = BC_0 \cap L^p$ under the norms $|x|_0$ and $|x|_Y = \frac{1}{2}(|x|_0 + |x|_p)$ respectively, and if we consider L as defined above on the domain $D_0^p(L)$

$= \{x \in BC_0 : x$ is absolutely continuous in $t \geqq t_0$ and $Lx \in BC_0 \cap L^p\}$, then setting $S = S_0^p = \{x \in D_0^p(L) : x(t_0) \in X_2^0\}$ it is not difficult to modify the argument for Theorem 1 to obtain the following.

THEOREM 2. *Suppose* H3 *and* $H4_0^p$ *hold where*

$H4_0^p$. *for each* $g(t) \in BC_0 \cap L^p$ *there exists at least one solution* $z(t) \in BC_0$ *of* (H).

If h satisfies the condition (2.1) *with* L^p *replaced by* $BC_0 \cap L^p$, *then the conclusions of Theorem* 1 *hold with* L^p *replaced by* $BC_0 \cap L^p$, *BC replaced by* BC_0, *and* P_1 *replaced by* P_1^0.

Remarks. (a) Inequality (2.1) is satisfied if for example $|h(t)(x) - h(t)(y)| \leqq \theta(t)|x - y|_0, t \geqq t_0$, where $\theta \in L^p$ or $\theta \in BC_0 \cap L^p$. Note also that the hypotheses on h are satisfied when h is "higher order" in x from BC to L^p (or $BC_0 \cap L^p$ to L^p); i.e., if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|h(t)(x) - h(t)(y)|_Y \leqq \varepsilon|x - y|_0$ for all $|x|_0, |y|_0 \leqq \delta$. In this case we simply take $r = \delta(\theta^0)$. Such perturbations appear frequently in the theory of differential, integral and integro-differential systems [1], [3], [6], [7], [9], [11]. As a simple illustration (applicable to our application below) h may have the form

$$h \equiv a(t, x) + \int_{t_0}^{t} k_1(t, s)b(s, x(s))\, ds + \int_{t_0}^{t_1} k_2(t, s)c(s, x(s))\, ds,$$

where $a(t, \xi)$, $k_1(t, \xi)$ and $k_2(t, \xi)$ are all $o(|\xi|)$ uniformly in $t \geqq t_0$ and

$$\sup_{t \geqq t_0} \int_{t_0}^{t} |k_1(t, s)|\, ds < +\infty, \qquad \sup_{t \geqq t_0} \int_{t_0}^{t_1} |k_2(t, s)|\, ds < +\infty.$$

(b) The spaces X_2 and X_2^0 (and hence S^p and S_0^p) are not uniquely determined (unless $X_1 = R^n$ or $X_1^0 = R^n$ in which case $X_2 = \{0\}$ or $X_2^0 = \{0\}$). It is to be noted that the constant c whose existence is asserted by Theorems 1 and 2 depends on X_2 or X_2^0 respectively.

Many recent papers [1], [6], [7], [9], [10], [11] (and the references therein) have dealt with stability properties of integro-differential and Volterra integral equations. Theorems 1 and 2 have implications about the stability or instability of (P). To point this out explicitly we make the following definitions for (P) (and its special case (NH) and (H)). System (P) (or more precisely the zero solution of (P) corresponding to $g \equiv 0$) is called *conditionally stable on* L^p if there exists a set of vectors $M \subseteq R^n$ whose closure contains the origin for which, to any $\varepsilon > 0$, there corresponds a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|g|_p \leqq \delta$, $g \in L^p$, and $x_0 \in M$, $|x_0| \leqq \delta$, implies the solution of (P) satisfying $x(t_0) = x_0$ exists for all $t \geqq t_0$ and satisfies $|x|_0 \leqq \varepsilon$. If, in addition to possessing conditional stability on L^p, (P) has the property that all solutions $x(t)$ corresponding to $x_0 \in M$, $|x_0| \leqq \delta_0$, and $|g|_p = \frac{1}{2}(|g|_p + |g|_0) \leqq \delta_0$, $g \in BC_0 \cap L^p$ for some fixed constant $\delta_0 > 0$ tend to zero as $t \to +\infty$, then (P) is called *conditionally asymptotically stable on* $BC_0 \cap L^p$. If M is an entire n-dimensional sphere in R^n, then (P) is called *stable* or *asymptotically stable* on the corresponding space. (These definitions of stability are special cases of more general definitions given for Volterra integral equations in [1].) System (P) is called *unstable on* L^p if it is not stable on L^p; i.e., if there exists a $\delta^* > 0$ and an $\varepsilon^* > 0$ such that for *every* initial vector x_0, $|x_0| \leqq \varepsilon^*$, and every

$g \in L^p$, $|g|_p \leqq \varepsilon^*$, the corresponding solution of (P) satisfies $|x(t)| > \varepsilon^*$ for some $t \geqq t_0$. Finally, we say that (H) *preserves* a given stability (or instability) property under the perturbation h if (P) has this stability (or instability) property. From Theorems 1 and 2 we can assert the following.

COROLLARY. *Under hypotheses* H3 *and* H4^p *system* (H) *preserves conditional stability, stability and instability on* L^p *for perturbations* h *satisfying the conditions of Theorem* 1. *Under* H3 *and* H$_0^p$, (H) *preserves conditional asymptotic stability and asymptotic stability on* $BC_0 \cap L^p$ *for* h *satisfying the conditions of Theorem* 2.

The case of stability and asymptotic stability preservation (which corresponds to the special case $X_1 = R^n$ or $X_1^0 = R^n$), under these conditions, is known (although proved and usually stated quite differently); see [1], [6], [11]. The preservation of instability and conditional stability is a generalization of known results for differential equations, where $B(t, s) = 0$ [3], [8].

3. The hypotheses H4^p and H4$_0^p$. We wish now to discuss briefly the assumptions H4^p and H4$_0^p$ for the linear system (NH) in order to give some insight into when they are fulfilled and how this can be determined. Further discussion of this question appears in § 4 below for the very important special case when $A(t) \equiv A$ = const. and $B(t, s) \equiv B(t - s)$. If H4^p holds, then given $g(t) \in L^p$, there exists a *unique* bounded solution $z(t)$ of (NH) satisfying $z_0 = z(t_0) \in X_2$; for, if z_1 and z_2 are two such solutions, then $y = z_1 - z_2$ is a bounded solution of (H) with $y(t_0) \in X_2$ and, hence, by the way X_1 and X_2 are defined, it follows that $y(t) \equiv 0$. This then establishes a function from L^p into $X_2 \in R^n$. The Corollary above in § 2 applied (with $h = 0$) to (NH) implies that this function is continuous. For $1 \leqq p < +\infty$ it follows easily from well-known theorems in functional analysis that there exists an $n \times n$ matrix $P(t)$, $|P(t)| \in L^q[t_0, +\infty)$, $q^{-1} + p^{-1} = 1$ for $p \neq 1$ and $q = +\infty$ for $p = 1$, such that

$$z_0 = -\int_{t_0}^{+\infty} P(s)q(s)\,ds. \tag{3.1}$$

The solution of (NH) is given by the variation of constants formula

$$z(t) = Y(t, t_0)z_0 + \int_{t_0}^{t} Y(t, s)g(s)\,ds, \qquad t \geqq t_0, \tag{VC}$$

where $Y(t, s)$ is the so-called fundamental solution matrix (or differential resolvent) of (NH); i.e., Y is the solution of the matrix equation

$$Y_t(t, s) = A(t)Y(t, s) + \int_{s}^{t} B(t, r)Y(r, s)\,dr, \qquad t \geqq s \geqq t_0,$$

$$Y(s, s) = I,$$

$I = n \times n$ identity matrix. This can be seen by straightforward substitution into (NH) (also see [6]). Thus, under H4^p, the unique bounded solution of (NH) with $z_0 \in X_2$ is given by

$$z(t) = -Y(t, t_0)\int_{t_0}^{+\infty} P(s)g(s)\,ds + \int_{t_0}^{t} Y(t, s)g(s)\,ds, \tag{3.2}$$

or

$$z(t) = \int_{t_0}^{t} V(t,s)g(s)\,ds + \int_{t}^{+\infty} W(t,s)g(s)\,ds \tag{3.3}$$

where

$$\begin{aligned} V(t,s) &= Y(t,s) - Y(t,t_0)P(s), && t_0 \leqq s \leqq t, \\ W(t,s) &= -Y(t,t_0)P(s), && t_0 \leqq t \leqq s. \end{aligned} \tag{3.4}$$

Using standard arguments (see for example [3], where the arguments used for differential equations carry over almost verbatim), one can show that (recall $p \neq +\infty$)

$$\int_{t_0}^{t} |V(t,s)|^q\,ds + \int_{t}^{+\infty} |W(t,s)|^q\,ds \leqq K, \qquad t \geqq t_0, \quad p \neq 1, \tag{3.5}$$

$$|V(t,s)| \leqq K \quad \text{for } t_0 \leqq s \leqq t \quad \text{and} \quad |W(t,s)| \leqq K \quad \text{for } t_0 \leqq t \leqq s, \quad p \neq 1, \tag{3.6}$$

for some constant $K > 0$.

It is the converse of this fact which interests us here. If a $P(t)$ can be found such that (3.5) or (3.6) holds, $1 \leqq q \leqq +\infty$, for W and V defined by (3.4), then given any $g(t) \in L^p$, $1 \leqq p \leqq +\infty$ (including now the case $p = +\infty$ corresponding to $q = 1$) it follows that the function $z(t)$ defined by (3.3) is a bounded solution of (NH) and, hence, $H4^p$ holds. That $z(t)$ is a solution follows from the fact that it can be rewritten in the form (3.2) and that it is bounded follows from a simple application of Hölder's inequality.

THEOREM 3. *If an $n \times n$ matrix $P(t)$ can be found such that* (3.5) *holds for some integer q, $1 \leqq q < +\infty$, for W and V defined by* (3.4), *then* $H4^p$ *holds for p such that $p^{-1} + q^{-1} = 1$ if $q \neq 1$ or $p = +\infty$ if $q = 1$. If* (3.6) *holds, then* $H4^p$ *holds for $p = 1$.*

In the differential equations case ($B \equiv 0$), it turns out that $P(t) = -P_2 Y^{-1}(t)$, where $Y(t)$ is a fundamental solution matrix. Since $Y(t,s) = Y(t)Y^{-1}(s)$ in this case, one easily finds that $W(t,s) = Y(t)P_2Y^{-1}(s)$ and $V(t,s) = Y(t)P_1Y^{-1}(s)$ (note: $P_1 + P_2 = I$). The conditions (3.5), (3.6) are, in this case, familiar in the study of bounded solutions [3], [8]. In the autonomous case $A(t) \equiv A$, the projections P_1 and P_2, roughly speaking, "select out" the eigenvalues of A with nonpositive and nonnegative real parts respectively. Thus, $H4^p$ for $p = 1$ is satisfied if those eigenvalues with zero real parts are simple; and, if all eigenvalues of A have nonzero real parts, then $H4^p$ holds for all $1 \leqq p \leqq +\infty$. In the next section § 4 we indicate how these features roughly carry over to (NH) in the case $B \neq 0$ but $B(t,s) \equiv B(t-s)$.

For the hypothesis $H4_0^p$ we have the following result.

THEOREM 4. *Suppose an $n \times n$ matrix $P(t)$ can be found such that, in addition to* (3.5) *for some q, $1 \leqq q < +\infty$, the condition*

$$\int_{t_0}^{T} |V(t,s)|\,ds \to 0 \quad \text{as } t \to +\infty \tag{3.7}$$

for every fixed $T \geqq t_0$. Then $H4_0^p$ *holds for p such that $p^{-1} + q^{-1} = 1$ if $q \neq 1$ or*

$p = +\infty$ *if* $q = 1$.

If (3.6) *holds and in addition*

$$|V(t, s)| \to 0 \quad \text{as } t \to +\infty \tag{3.8}$$

for every $s \geqq t_0$, *then* $H4_0^p$ *holds for* $p = 1$.

Proof. If (3.5) holds for $q \neq 1$, then $z(t)$ defined by (3.3) (or by (3.2)) is a bounded solution of (NH), as pointed out above, for each $g \in BC_0 \cap L^p$. We want to show in addition that $|z(t)| \to 0$ as $t \to +\infty$. Given any $\varepsilon > 0$ choose $T = T(\varepsilon) > 0$ so large that $(\int_T^{+\infty} |g|^p \, ds)^{1/p} \leqq \varepsilon/2K$. Then from

$$z(t) = \int_{t_0}^{T} V(t, s)g(s)\, ds + \int_{T}^{t} V(t, s)g(s)\, ds + \int_{t}^{+\infty} W(t, s)g(s)\, ds$$

for $t \geqq T$, follows (after an application of Hölder's inequality and (3.6)) that

$$|z(t)| \leqq \int_{t_0}^{T} |V(t, s)|\, ds\, |g|_0 + \varepsilon$$

which implies, upon letting $t \to +\infty$, that $\limsup_{t \to +\infty} |z(t)| \leqq \varepsilon$. Inasmuch as $\varepsilon > 0$ was arbitrary it follows that $|z(t)| \to 0$ as $t \to +\infty$. The case $p = +\infty$ is similar.

If (3.6) holds, then we have

$$|z(t)| \leqq \int_{t_0}^{T} |V(t, s)|\, |g(s)|\, ds + \varepsilon,$$

where $T \geqq t_0$ is chosen so that $\int_T^{+\infty} |g|\, ds \leqq \varepsilon/2K$. Using the dominated convergence theorem and (3.8) we again conclude that $|z(t)| \to 0$ as $t \to +\infty$. □

4. The convolution case and an application. Suppose $A(t) \equiv A = \text{const.}$ and $B(t, s) \equiv B(t - s)$ in (NH). Take $t_0 = 0$. Most applications of (P) have this form so it is important to develop techniques for testing $H4^p$ and $H4_0^p$ in this case; specifically, we wish to determine some conditions on A and B under which $H4^p$ and $H4_0^p$ hold for $p = 1$ and $+\infty$. In this case $Y(t, s) = Y(t - s)$, where $Y(t)$ solves the matrix equation

$$\begin{aligned} Y'(t) &= AY(t) + \int_0^t B(t - r)Y(r)\, dr, \\ Y(0) &= I. \end{aligned} \tag{4.1}$$

We assume from now on that $B \in L^1[0, +\infty)$. Letting * denote the Laplace transform, we have from (4.1) the equation

$$(sI - AB^*(s))Y^*(s) = I \tag{4.2}$$

for Y^*. A straightforward application of Gronwall's lemma to the equivalent integral equation for (4.1) together with the assumption $B \in L^1$ implies Y is exponentially bounded. A necessary and sufficient condition for what Miller [7], [9], [11] calls uniform asymptotic stability of (NH) is that

$$p(s) \equiv \det(sI - A - B^*(s)) \neq 0 \quad \text{for } \operatorname{Re} s > 0. \tag{4.3}$$

In terms of our hypotheses above, this result supplies necessary and sufficient conditions for the stability of (NH) on $L^\infty = BC$ (i.e., for the validity of hypothesis $H4^\infty$ in its strongest form: *all* solutions are bounded for each choice of g). This is because, as shown in [7], this condition implies $Y \in L^1$ and hence the stability on BC (see (VC)). For conditional stability and instability we expect the eigenvalue-like condition (4.2) to be relaxed in such a way as to allow for roots in the right half-plane.

Let us suppose then that $p(s)$ has roots in the right half-plane. We do not intend to study this situation in depth here, but instead to restrict our attention to remarks appropriate to our application below. See [13] for a more extensive study of this problem. Let r be any root of $p(s)$. We say that r has *algebraic multiplicity* $\mu \geqq 1$ if $p^{(i)}(r) = 0$ for $i = 0, 1, \cdots, \mu - 1, p^{(\mu)}(r) \neq 0$, and has *geometric multiplicity* $m \geqq 1$ if the $n \times n$ matrix $sI - A - B^*(s)$ at $s = r$ has rank $n - m$.

Let k, $0 \leqq k < +\infty$, be the number of roots of $p(s)$ such that $\operatorname{Re} s \geqq 0$. For simplicity assume all of these roots r_p^+, $1 \leqq p \leqq k$, have algebraic multiplicity $\mu = 1$. Let r_p^-, $p = 1, 2, \cdots$, denote the remaining roots (which may be infinite in number); $\operatorname{Re} r_p^- < 0$. Set $Y(t) = [y_{ij}(t)]$, $e_i = \operatorname{col}(\delta_{ij})$, and $p_{ij}(s)$ equal to the cofactor of the ijth entry of the matrix $sI - A - B^*(s)$. Solving (4.2) for the jth column of $Y^*(s)$ using Cramer's rule, we find $y_{ij}^*(s) = p_{ij}(s)/p(s)$, $1 \leqq i, j \leqq n$. We now assume (i) $B^*(s)$ is meromorphic in the entire complex plane and (ii) the estimate $|p_{ij}(s)/p(s)| \leqq K/|s|^\alpha$ holds for some constants K, $\alpha > 0$ and all s, $\operatorname{Re} s \geqq s_0 \geqq \max\{\operatorname{Re} r_p^+\}$. Since Y and hence y_{ij} are exponentially bounded, the complex inversion formula of p_{ij}/p along $\operatorname{Re} s = s_0$ exists and represents $Y(t)$, $t > 0$ [2, p. 183]. Also (ii) guarantees the validity of the residue series expansion for $y_{ij}(t)$ [2, p. 193]. Thus, if $\rho_{ij}(r_p^\pm)$ is the residue of $e^{st}p_{ij}(s)/p(s)$ at $s = r_p^\pm$ (for the simple roots $s = r_p^+$, $\rho_{ij}(r_p^+) = e^{r_p^+ t}p_{ij}(r_p^+)/p'(r_p^+)$), then

$$(4.4) \qquad y_{ij}(t) = \sum_{p=1}^{\infty} \rho_{ij}(r_p^-) + \sum_{p=1}^{k} p_{ij}(r_p^+)\, e^{r_p^+ t}/p'(r_p^+), \qquad t > 0,$$

for $1 \leqq i, j \leqq n$. The residues $\rho_{ij}(r_p^-)$ are all of the order $t^{\mu_p - 1} e^{r_p^- t}$. We can write then $Y(t) = Y^-(t) + Y^+(t)$, where $Y^-(t) = \sum_{p=1}^\infty \rho_{ij}(r_p^-)$ and $Y^+(t) = \sum_{p=1}^k \gamma_{ij}^p e^{r_p^+ t}$ with $\gamma_{ij}^p = p_{ij}(r_p^+)/p'(r_p^+)$.

Now we wish to construct $P(s)$ as in Theorem 3. Referring to (3.4) we have

$$V(t, s) = [Y^-(t - s) - Y^-(t)P(s)] + [Y^+(t - s) - Y^+(t)P(s)],$$
$$W(t, s) = -[Y^-(t)P(s) + Y^+(t)P(s)].$$

If we choose P such that

$$(4.5) \qquad Y^+(t - s) - Y^+(t)P(s) = 0$$

(and only if we do this), then as we will point out below, V and W will satisfy (3.5) with $q = 1$ if none of the r_p^+ lie on the imaginary axis and (3.6) in case some r_p^+ are on the imaginary axis.

To solve (4.5) for P we first consider the equation

$$(4.6) \qquad [\gamma_{ij}^p]P(s) = [\gamma_{ij}^p]\, e^{-r_p^+ s}$$

for fixed p, $1 \leqq p \leqq k$. Now $sI - AB^*(s)$ is singular at $s = r_p^+$ which implies that the sum of the products of the cofactors of any row times the corresponding elements of any row is zero. This means all rows of $r_p^+ I - A - B^*(r_p^+)$ are in the kernel of $[\gamma_{ij}^p]$ which in turn implies that the nullity of $[\gamma_{ij}^p]$ is greater than or equal to the rank, $n - m_p$, of $r_p^+ I - A - B^*(r_p^+)$, where $m_p \geqq 0$ is the geometric multiplicity of r_p^+. Thus, $rk[\gamma_{ij}^p] = n -$ nullity $\leqq n - (n - m_p) = m_p$. Assume now that all roots r_p^+ in the right half-plane also have geometric multiplicity one: $m_p = 1$ for all $p = 1, \cdots, k$. Then $rk[\gamma_{ij}^p] \leqq 1$. It is obvious in (4.6) that $[\gamma_{ij}^p]$ and the augmented matrix $[\gamma_{ij}^p | \gamma_{ij}^p e^{-r_p^+ s}]$ have the same rank. Thus, if $p_q(s)$ is the qth column of $P(s)$, we can eliminate all but one equation for each p from (4.6) and obtain a $k \times n$ system to be solved for p_q. This can be done for every column $1 \leqq q \leqq n$ of $P(s)$. Notice that $P(s)$ is a linear combination of exponents $e^{-r_p^+ s}$ and thus $P(s) \in BC$ or L^1 depending on whether or not $\operatorname{Re} r_p^+ = 0$ for some p. Moreover we have

$$V(t,s) = Y^-(t-s) - Y^-(t)P(s), \qquad W(t,s) = -[Y^-(t)P(s) + Y^+(t)P(s)],$$

and it is easily seen from the properties of $P(s)$ and Y^- that (3.6) (or (3.5) with $q = 1$) holds depending on whether $\operatorname{Re} r_p^+ = 0$ for some p (or not).

As an application of this approach and the perturbation Theorems 1 and 2 we consider the system

$$N_1' = N_1(e_1 - \gamma_1 N_2),$$

$$N_2' = N_2\left(-e_2 + \gamma_2 N_1 + \int_{-\infty}^{t} b(t-s) N_1(s)\, ds\right),$$

where $e_i > 0, \gamma_1 > 0$ and $\gamma_2 \geqq 0$ are constants and where $b(t) \geqq 0, b \in L^1[0, +\infty)$. This system is Volterra's model of a predator-prey population with hereditary effects [14, Chap. 4]. Besides initial conditions for N_1 and N_2 at $t_0 = 0$, it is assumed that N_1 is known for $t \in (-\infty, 0]$; say $N_1(t) = N_1^0(t)$. Defining $K_1 = e_1/\gamma_1$, $K_2 = e_2(\gamma_2 + |b|_1)^{-1}$, $x_1 = \log(N_1/K_1)$, $x_2 = \log(N_2/K_2)$ and $x = \operatorname{col}(x_1, x_2)$, we find that this system transforms to (P) with

$$A = \begin{bmatrix} 0 & -\gamma_1 K_2 \\ \gamma_2 K_1 & 0 \end{bmatrix}, \qquad B(t-s) = \begin{bmatrix} 0 & 0 \\ K_1 b(t-s) & 0 \end{bmatrix},$$

$$h(t)(x) = \begin{bmatrix} -\gamma_1 K_2(q(x_2) - x_2) \\ \gamma_2 K_1(q(x_1) - x_1) + K_1 \int_0^t b(t-s)(q(x_1) - x_1)\, ds \end{bmatrix},$$

$$g(t) = \begin{bmatrix} 0 \\ K_1 \int_{-\infty}^{0} b(t-s) q(N_1^0)\, ds \end{bmatrix},$$

where $q(x) = e^x - 1$. Observe that $|h(t)(x)| \leqq L|x|_0^2$ for all $x \in BC$ and some constant $L > 0$. Thus, h maps $\Sigma(r) \cap L^\infty$ $(= BC)$ into BC in such a way that (2.1) holds for small θ $(\leqq \theta^0)$ provided r is small; that is, the hypotheses of Theorems 1 and 2 for the perturbation term h are fulfilled with $p = +\infty$. (Note that $|g|_0$ is small if $|N_1^0|_0$ is small since $b \in L^1$.) Thus, to draw the conclusions of Theorems 1 and 2 and the Corollary we need only verify H4′ and H4$_0^\circ$ respectively (H3 is obviously fulfilled). This we will do by using Lemma 2 above. We will only consider the following case: $b(t) = (\alpha t + \beta) e^{-\delta t}$ for $\alpha \geqq 0, \beta \geqq 0, \delta > 0$ and $\alpha^2 + \beta^2 \neq 0$. The ecological interpretation of $b(t)$ can be found in Volterra's original work [14]: $b(t) = b_1(t)b_2(t)$, where $b_1(t)$ is the probability of a predator born at time $t = 0$ surviving to time t and $b_2(t)$ is the expected number of offspring (per predator) per unit time born to the population of predators at time t per unit encounter with prey at time $t = 0$. The case $\alpha = \delta = 0$ was considered numerically by Davies in [5], where instability was found. However, this case (where $b(t) \equiv$ const.) is not particularly realistic in view of the physical interpretation of $b(t)$ (and also not amenable to our analysis since we demand, as does Volterra, that $b \in L^1$). The case $\alpha = 0$, $\delta > 0$ falls into the case of monotonically non-increasing kernels $b(t)$ considered by Miller [11]; however, his results appear to have a mistake (Corollary 3 on p. 264 seems to be false) and are in fact contradicted by our results below. This case is more reasonable than the one considered by Davies in that the hereditary effects represented by $b(t)$ decrease monotonically with time. Perhaps an even more reasonable case is $\alpha \neq 0$, where the full measure of the hereditary effects on predator births due to past encounters with prey are not instantaneously felt, but gradually increase to a maximum before decreasing monotonically to zero with time.

Proceeding as above, we must investigate the matrix $sI - A - B^*(s)$ which in this application is

$$\begin{bmatrix} s & \gamma_1 K_2 \\ -\gamma_2 K_1 - K_1 b^*(s) & s \end{bmatrix},$$

where $b^*(s) = (\beta s + \alpha + \beta\delta)/(s + \delta)^2$. Obviously $b \in L^1$ and $b^*(s)$ is meromorphic in the complex s-plane. Now $p(s) = s^2 + a_1 a_2 + a_2 b^*(s)$ or

$$p(s) = \frac{n(s)}{(s + \delta)^2}, \tag{4.7}$$

$$n(s) = s^4 + 2\delta s^3 + (\delta^2 + a_1 a_2)s^2 + (2\delta a_1 a_2 + a_2\beta)s + \delta^2 a_1 a_2 + a_2(\alpha + \beta\delta),$$

where $a_1 = \gamma_2 \geqq 0$, $a_2 = \gamma_1 K_1 K_2 > 0$. It is not difficult to check that condition (ii) above holds (with $\alpha = 1$). Thus, we have only to investigate the roots of $p(s)$ lying in the right half-plane; these roots coincide with those of the numerator $n(s)$ in (4.7). An application of the Hurwitz criteria to $n(s)$ shows that not all roots lie in the left half-plane (the third Hurwitzian determinant is negative for $\beta^2 + \alpha^2 \neq 0$ and zero for $\beta = \alpha = 0$). Moreover, making the substitution $\bar{s} = -s$ in $n(s)$ we find that $n(\bar{s})$ also cannot have all of its roots in the left half-plane (since the coefficient of $\bar{s}^3$ is $-2\delta < 0$); i.e., $n(s)$ cannot have all of its roots in the right half-plane. Finally it is easy to check that $n(s)$ has no roots on the imaginary axis nor on the positive real axis. Hence, we conclude that $n(s)$ has two roots in the left half-plane and $k = 2$ complex conjugate roots in the right half-plane both of algebraic multiplicity one. Thus, $p(s)$ has two such roots in the right half-plane. Whether the roots of n in the left half-plane are roots of p depends on whether either of these roots equals $-\delta$ or not. It is not difficult to investigate $n(s)$ at $s = -\delta$. We find that (a) $p(s)$ has two conjugate roots ($\neq \delta$) in the left half-plane if $\alpha \neq 0$ or (b) $p(s)$ has one negative root of multiplicity one ($\neq \delta$) if $\alpha = 0$. Since $n = 2$ and $sI - A - B^*(s)$ at any of these roots is singular but not identically zero, the geometric multiplicity of all (and in particular the two roots in the right half-plane) is also one. One can verify that the system (4.6) reduced by elimination of all but one equation for each p is solable for the matrix P. Hence, the linearized, nonhomogeneous system for this example satisfies both $H4^\infty$ and $H4_0^\infty$. This means, by the preservation Theorems 1 and 2, that the *nonlinear Volterra model above preserves the instability of its linearized system.* To be more specific about this instability we must determine X_1. The general solution of the linearized system is $Y(t)y_0$, where $Y(t)$ is given by (4.4), $n = 2$. Clearly, y_0 can be chosen so that $Y(t)y_0 \in BC$ or BC_0 if and only if $[p_{ij}(r_1^+)]y_0 = [p_{ij}(r_2^+)]y_0 = 0$. Recalling the definition of p_{ij} and that $r_2^+ = \bar{r}_1^+$ in our example and referring to A above, we can show without difficulty that the only solution to these simultaneous systems is $y_0 = 0$; i.e., $X_1 = \{0\}$ and no solutions of the linear system exist in BC or BC_0 (except the zero solution). This means (cf. Theorem 1) that there exist constants r, b', and $a > 0$ such that for any $g \in BC$, $|g|_0 \leqq a$, all solutions of the Volterra system (except $x \equiv 0$) satisfying $|x(t_0)| \leqq b'$ must satisfy $|x(t)| > r$ for some $t > t_0$. Or, in other words, if the initial size of the prey population $N_1^0(t)$ is small enough, then no solution $N_1(t)$, $N_2(t)$ exists to Volterra's model which for all t remain close to the "critical points" K_1, K_2 respectively, no matter how close the initial populations $N_1(0)$, $N_2(0)$ are taken to K_1, K_2 respectively.

REFERENCES

[1] J. M. BOWNDS AND J. M. CUSHING, *On preserving stability of Volterra integral equations under a general class of perturbations*, to appear.

[2] R. V. CHURCHILL, *Operational Mathematics*, McGraw-Hill, New York, 1958.

[3] W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath, Boston, Mass., 1965.

[4] C. CORDUNEANU, *Some perturbation problems in the theory of integral equations*, Math. Systems Theory, 1 (1967), pp. 143–155.

[5] H. T. DAVIS, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, 1962.

[6] S. I. GROSSMAN AND R. K. MILLER, *Perturbation theory for Volterra integrodifferential systems*, J. Differential Equations, 8 (1970), pp. 457–474.

[7] ———, *Nonlinear Volterra integrodifferential systems with L^1-kernels*, Ibid., 13 (1973), pp. 551–566.

[8] J. L. MASSERA AND J. J. SCHÄFFER, *Linear differential equations and functional analysis, I*, Ann. of Math., 67 (1958), pp. 517–573.

[9] R. K. MILLER, *Asymptotic stability properties of linear Volterra integrodifferential equations*, J. Differential Equations, 10 (1970), pp. 485–506.

[10] ———, *Admissibility and nonlinear Volterra integral equations*, Proc. Amer. Math. Soc., 25 (1970), pp. 65–71.

[11] ———, *Asymptotic stability and perturbations for linear Volterra integrodifferential systems*, Delay and Functional Differential Equations and their Applications, K. Schmitt, ed., Academic Press, New York, 1972.

[12] ———, *Nonlinear Volterra Integral Equations*, Benjamin Press, Menlo Park, Calif., 1971.

[13] ———, *Structure of solutions of unstable linear Volterra integrodifferential equations*, J. Differential Equations, 15 (1974), pp. 129–157.

[14] V. VOLTERRA, *Lecons sur la Théorie Mathématique de la Lutte par la Vie*, Gauthier-Villars, Paris, 1931.