The Arizona Experience Software Development & Use[1]

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Mathematics faculty members at the University of Arizona are involved with several aspects of curriculum reform which heavily rely on computer software. All the software we use has been developed by us and is designed so that neither the instructor nor the student need be "computer literate". This software:

- aids students in solving problems
- encourages students to treat mathematics as an experimental subject
- captures the spirit and excitement of current developments in mathematics and science
- helps the instructors and students during lectures and examinations
- reviews prerequisite material and identifies weak areas in a student's background.

To date we have released thirty-five pieces of software for MS-DOS machines. This software ranges from exploring one-dimensional iterative maps (period doubling, chaos) to the graphing of Fourier partial sums, from the curve-fitting of data by eye, to the plotting of direction fields and integral curves. The software may be classified under three categories: "Are You Ready?", "Slide Shows", and "Toolkits". All our software, including a user's manual for the toolkits [1], is available from our department for the cost of their reproduction plus the cost of mailing. It may be freely copied and distributed by anyone, but not sold for profit. (Software development is an ongoing project. A list of software currently available is given in the Appendix.)

The purpose of the "Are You Ready?" disks is to make available to students a computer program which reviews those materials from the prerequisite courses that are essential for success in the present course. In addition, the programs identify a student's weak areas and recommend appropriate action (usually references are to Schaum's Outlines because they are inexpensive and are not revised every few years). The usual use of these disks is for an instructor to announce on the first day of class that an hour exam over prerequisite material will be given one week later. The students are told that the questions on the exam will be exactly like those contained in the appropriate "Are You Ready?" disk. This practice quickly gets the attention of unprepared students and helps reacquaint them with prerequisite material before weaknesses impede the learning of new material. There are two additional benefits: first it forces the students to start working immediately in the class, and second all future exams are out of step with exams given in the students' other courses. Five "Are You Ready?" disks are available for courses ranging from College Algebra to Ordinary Differential Equations. Students may use these disks in our computer laboratory, or make a free copy for themselves and use machines elsewhere.

The ten "Slide Shows" are collections of screen images, primarily graphical, which anyone can view. These are usually images of functions which would be difficult, or impossible, to draw on the board. Some of these are animated and some "zoom" in on the function. These are used in the classroom to illustrate the lecture or used by students in our computer laboratory in connection with homework exercises.

The twenty disks under the category of "Toolkits" are used by both students and instructors, in and out of class. All have drop-down menus and are self documenting, with online, context sensitive help. Some of them allow you to "zoom" in on a function and others allow you to plot a function of one indepen-

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dent variable and three parameters, which then may be "twiddled" or "tuned" to show the effect of parameter variation. Each toolkit is developed with a single mathematical topic in mind. They all have a common interface and may be operated with a moment's instruction. These form an extremely useful (sometimes essential) resource for students in doing their homework.

The effects of this computer software on four mathematics courses will be mentioned now.

Calculus

The first two semesters of our calculus sequence (single variable) are being revised in accordance with ideas presented in [2]. We have developed a number of supplemental calculus projects with many goals in mind, some of which are to:

- give non-template type problems that require some reasoning
- have problems that require more than one step
- give the student experience when to use technology and when to use pencil and paper
- have a student make a conjecture and attempt to prove it
- give a student more practice at translating between words, diagrams, and mathematical symbols
- encourage the rule of three: analyze problems three ways; analytically, numerically, graphically
- present projects whose solutions require techniques from various parts of the course.

Projects can be simple or more extensive and often cover material from more than one course. A number of supplemental exercises for calculus, ranging from graphical problems, similar to those published by Peter Taylor [4], to those on vector calculus have been developed. Some of them are of an exploratory nature and lead to nonstandard applications of Riemann sums.

These projects are usually used for homework exercises but some instructors also use them as a "think along" presentation in the classroom. Two such presentations will be mentioned here. The first illustrates the evaluation of a derivative at a point. Each student in the classroom was given a handout which contains 24 pictures of a cat which document (in frames taken every .031 second) its progression from a walk to a gallop [3]. There was a uniform grid in the background of each picture so as a class we developed a table of distance moved (say of the cat's nose) as a function of time. This is presented in columns 1, 2, 3, and 4 of the table. Then the students had to develop a method to give accurately the velocity of the cat at the time corresponding to the twelfth frame. The method developed was to take the ratio of the difference of displacement of the cat divided by the time difference for each frame in comparison to the twelfth frame. After several calculations verified the feasibility of this approach the instructor provided the rest of the table below.

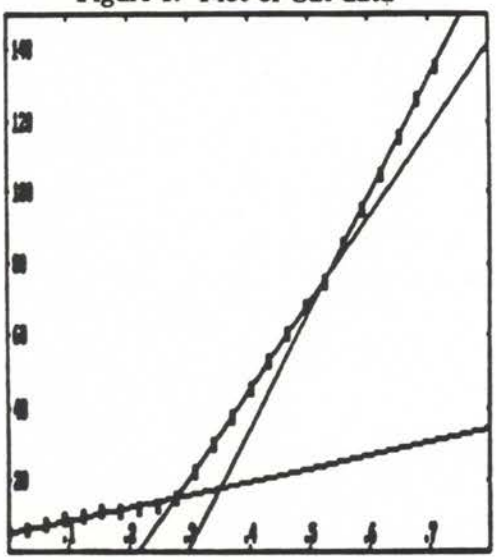
One thing the students immediately noted from the data in the table was the fact that from frames 10 through 18 the cat was moving with a constant velocity. The next thing we did was to enter the time and distance data as the x and y coordinates on a graph using the software package TWIDDLE. Displaying the graph of these data (shown in the figure) led the students to conclude that the cat's motion could be split into three different regimes (frames 1-9, 10-18, and 19- 24). Fitting three different straight lines to this data allowed them to compute the instantaneous velocity (the slope of each line) for the corresponding three time intervals. The class spent some time discussing the meaning of the "discontinuities" in the slope between the three different time periods, i.e. was it a result of the cat's movement or our model of the cat's movement. Another fact that required discussion was the apparent "dip" in the graph at the ninth frame.

A second presentation uses the software package FORTUNE. Here two functions were entered, $f(x) = b\sin(x) + c\cos(x)$ and $g(x) = [\sin(x+a) - \sin(x)]/a$. Initial values of $b = c = 0$ and $a = 1$ were also entered. Then the graph of g was shown for $a = 1$, .5, .1, .01, and .001. The students clearly saw that the graphs for the last two values of a were identical. The curves for values of a other than .001 were then deleted and the graph of f for $b = 0$ and $c = 1$ was drawn, coinciding with the graph of the difference quotient for g. This gave a dramatic illustration that the derivative of $\sin(x)$ is $\cos(x)$. Then we plotted the graph of f with $b = 1$ and $c = 0$, along with that of its derivative (g with $a = .001$). Having both curves on the screen allowed us to note that the slope of f at a specific value of x gave the value of $g(x)$. The slope was easily determined after a square grid was superimposed on the screen. The next thing we did with this display was to pick any point on the graph of f and note that by "zooming"

Table 1: Cat data

Photo	Time	Distance	Cms	Cms/Sec
1	0.000	1.0	5.0	73.31
2	0.031	1.2	6.0	77.42
3	0.062	1.5	7.5	80.65
4	0.093	1.8	9.0	84.68
5	0.124	2.0	10.0	92.17
6	0.155	2.2	11.0	102.15
7	0.186	2.3	11.5	119.35
8	0.217	2.4	12.0	145.16
9	0.248	2.5	12.5	188.17
10	0.279	3.0	15.0	241.94
11	0.310	4.5	22.5	241.94
12	0.341	6.0	30.0	
13	0.372	7.5	37.5	241.94
14	0.403	9.0	45.0	241.94
15	0.434	10.5	52.5	241.94
16	0.465	12.0	60.0	241.94
17	0.496	13.5	67.5	241.94
18	0.527	15.0	75.0	241.94
19	0.558	17.0	85.0	253.46
20	0.589	19.0	95.0	262.10
21	0.620	21.0	105.0	268.82
22	0.651	23.0	115.0	274.19
23	0.682	25.2	126.0	281.52
24	0.713	27.0	135.0	282.26

Figure 1: Plot of Cat data



in, the graph would eventually appear as a straight line. Later on in the course we would reflect (on the computer screen) the graph of a function f about the line $y = x$ to show the relationship between f and the inverse of f.

We use software to help the calculus student understand the difference between global behavior of a continuous function (such as why all polynomials of odd degree look alike when seen from afar) and local behavior (all differentiable functions look linear up close). We also use software, including graphical displays, to suggest the proper reasoning and equation manipulation needed to verify what the screen suggests, or prove it is wrong. Having a software package to generate a direction field instantly allows us to use implicit differentiation to plot the graph of an implicit relation. For example, SLOPES was used to determine the graph of $\sin(x^2 + y^2) + \cos^2(xy) = C$, for various values of C. Also having a software package, ROOTFIND, that easily finds roots of functions means that all sorts of "real world" examples may be used instead of those that lead to factoring a binomial with rational roots.

Ordinary Differential Equations

In the elementary ordinary differential equations course, we have prepared a supplement on direction fields to be used in conjunction with our software. We started the course with this topic and used direction fields to motivate what type of analytical solution to seek in some special cases where closed form solutions exist. For example, before doing anything else with the equation $y' = y$, we looked at the direction field, which suggested we seek an exponential solution. Also after introducing the logistic equation $y' = ay(b - y)$, with the intent to "level off" the unrealistic exponential growth for all time, we immediately checked the direction field to see if this model succeeded in that respect. The direction field motivated the conjecture that the asymptotic limit was b, as well as the concept of an equilibrium solution. It was easy to check the direction field for a Gompertz growth law to see whether it was more like exponential growth or logistic. The same approach was used for second order differential equations where spring-mass problems were first treated in the phase plane and conjectures were made (regarding oscillations, damped motion, undamped motion, resonance, etc.). Only then did we use analytical methods to verify the conjectures and extend the analysis.

We have a software package OLDES which allows you to enter a recurrence relation and values of the first two coefficients to obtain an analytical finite sum, and/or graph, for a power series solution of a second-order linear differential equation. Seeing the partial sums build up graphically drives home the point that a power series solution is really a function. (This point was usually missed by students seeing a standard approach.) We can also compare the first few terms of the Taylor series solution of a nonlinear problem with a power series solution of its linear approximation. The program has been modified to cover Frobenius type solutions.

First-order systems of linear differential equations are solved by finding the eigenvalues and eigenfunctions needed to develop a fundamental matrix. The characteristic equation, its roots and corresponding eigenfunctions are found using the software package LINALG. Again more realistic problems (without having to have integer values for the eigenvalues) may be considered.

Some of the "Slide Shows" are used in this course to illustrate what can go wrong with numerical solution of simple differential equations. Others are of value in illustrating mathematical models including wave phenomena.

The toolkit FOURIER allows you to enter a function (and its Fourier coeffficients if you desire) and then compare the periodic extension of this function with partial sums of its Fourier Series. Gibbs phenomena are particularly striking with this display.

The toolkit OLDES was used to display behavior of analytical solutions of initial value problems. For example, damped motion and beat phenomena become much clearer using this package. More importantly, by working the exercises, students may discover many of the pertinent results themselves. We can also compare an analytical solution to a numerical solution of the appropriate initial value problem, using RK4 with an adaptive step size.

We feel that this approach makes differential equations much more of an intuitive course, which avoids the usual cookbook approach. We are currently developing a series of expanded exercises on ordinary differential equations in keeping with the objectives listed above.

Elementary Linear Algebra

In the linear algebra course students make extensive use of the toolkit LINALG for problems involving vectors, matrices, and systems of linear equations. This makes possible the assignment of more substantial problems and allows for more realistic applications of the general theory. It is impressed upon the students, of course, that having this calculating tool is by no means a substitute for learning precisely how the algorithms work. Surprising as it may seem, using software which allows the student to give step by step commands and watch the correct result follow (in contrast to usual pencil and paper world) gives the student greater insight into what is going on. For example in solving systems of equations using a step by step reduction to an echelon form, the LINALG user has the standard elementary row operations available: Add a multiple of one row to another, Multiply a row by a scalar, Interchange two rows, Undo last operation. For all but the last choice, the choice of rows to be involved as well as the desired scalar must be entered by the student. The important thing is that the arithmetic operations are performed correctly and instantly. Thus the student can concentrate on new mathematics and not old arithmetic.

Typically students are required to hand in a computer homework asignment once each week. They are encouraged to use the computer when it is helpful in working on their routine daily homework problems, and indeed some of the daily problems are also designed with our computer routines in mind. The class is held in a computerized classroom, in which every student has a terminal, and they are allowed (and encouraged) to use LINALG as a tool while taking examinations. We have developed a series of short exploratory exercises which allow the student easily to work enough special cases, using LINALG, that a conjecture as to the form of a general rule can be made. After consulting with the instructor to verify the conjecture, the student is then sent away to prove the proposed result.

A list of options currently available in LINALG follows. Inner Product (+ Length, Angle, etc.);

Cross Product; Linear Independence; The Gram-Schmidt Process; Elementary Row Operations; Reduced Echelon Form; Determinant; Sums, Differences, and Scalar Multiples of Vectors and Matrices; Matrix Multiplication; Matrix Powers; Matrix Inversion; Similarity Transformations; Solve an Arbitrary Linear System; Least Squares Solutions; Characteristic Polynomials; The Power Method for Eigenvalues.

Course For Secondary School Teachers

The course entitled "An Introduction to Modern Algebra and Number Theory" is required for our secondary education majors. Two packages, COMPLEX NUMBERS and DIVISION ALGORITHM, were written with this course in mind. (Two others, INTERPOLATION and TWIDDLE, are also used.) The COMPLEX NUMBERS toolkit allows you to view any complex number you enter and to evaluate sums, products, or quotients of complex numbers. You may find the n^{th} roots of any complex number and see them displayed graphically. A series of "games" allows students to practice their skill on the graphical result of complex addition, subtraction, inversion, division, conjugation, or multiplication. The purpose of this software package is to allow students to explore the relationship between the algebraic and geometric aspects of complex numbers.

DIVISION ALGORITHM carries out algorithms such as Euclid's algorithm for finding the greatest common denominator of two polynomials and Sturm's algorithm for separating the roots of a polynomial. Without this package, these algorithms are messy; it is realistic to do only trivial examples by hand. (In fact in checking this package, we discovered several mistakes in examples given in current textbooks.) A nice feature of DIVISION ALGORITHM is that while the coefficients of your original polynomial (or polynomials) must be entered, all resulting polynomials may be saved and manipulated with only a few keystrokes.

INTERPOLATION is the toolkit used to allow students to see the effect of fitting a polynomial of degree n through $n+1$ data points (Lagrange Interpolation). All students need to do is enter the data set and the software does the calculation and draws the resulting graph. A press of an appropriate key allows comparison with the result of connecting the data points with one of three other types of approximation; linear, quadratic or cubic splines. Using the software allows one to focus on the phenomena and not the (in the usual case) messy details.

References

1. Lomen, David and David Lovelock (1991), *A Manual for the Mathematical Toolkit Software developed by the University of Arizona*, University of Arizona, Tucson, (16 pages).

2. Lovelock, David and Alan C. Newell (1987), *A Calculus Curriculum for the Nineties*, in L. A. Steen (Ed.), *Calculus for a new century: A pump not a filter*, MAA Notes, Number 8, Mathematical Association of American, Washington, DC (pp. 26-32).

3. Maybridge, Eadward (1957), *Animals in Motion*, Dover, New York (plate 124).

4. Taylor, Peter D. (1989), *An Introduction to the Analysis of Functions*, Queens University, Kingston, Ontario (pp. 1.1.1-1.14.6).

Appendix: Software Currently Available

ARE YOU READY?

ARE YOU READY FOR COLLEGE ALGEBRA? Reviews those parts of intermediate algebra which are essential for success in college algebra.

ARE YOU READY FOR CALCULUS? Contains *Are You Ready For Calculus I?* and *Are You Ready For Business Calculus?* Reviews those parts of college algebra (and trigonometry, in the case of calculus I), which are essential for success in calculus.

ARE YOU READY FOR CALCULUS II? Reviews those parts of precalculus and calculus I, which are essential for success in calculus II.

ARE YOU READY FOR CALCULUS III? Reviews those parts of calculus I and II, which are essential for success in calculus III.

ARE YOU READY FOR ODES? Reviews those parts of exponentials, logs, differentiation, integration, and power series, which are essential for success in ordinary differential equations.

SLIDE SHOWS

FUNCTIONS. This consists of graphs of functions which occur frequently in Calculus, such as $\sin(\frac{1}{x})$, $x\sin(\frac{1}{x})$, $\frac{\sin x}{x}$, $\frac{1-\cos x}{x}$, a^x and its derivative for various values of a, $\sin(2\pi x)+\sin(2\pi a x)$ for various values of a, and, that old standby, an everywhere continuous nowhere differentiable function.

TROUBLE. This demonstrates the dangers associated with graphing a function by plotting "enough" points and then joining them.

NEWTON'S METHOD. This shows Newton's method graphically. Some images demonstrate how it works, while others display how it can go wrong.

TAYLOR SERIES (SLIDE SHOW). This displays the Taylor polynomials, of various degrees, corresponding to $\exp x$, $\sin x$, $\cos x$, $\frac{1}{1-x}$, $\arctan x$, $\sqrt{1+x}$, and $\log(1+x)$. . It also contains some visual demonstrations of convergence and divergence.

ORDINARY DIFFERENTIAL EQUATIONS. This has examples from ordinary differential equations, including a one parameter family of curves, the US Population and logistic growth, the cooling of coffee and Newton's law, the Euler and Runge Kutta methods and their fallibility, damped free vibrations, building a function from its series solution, and Bessel functions.

FOURIER SERIES (SLIDE SHOW). This shows the partial sums of Fourier series for the triangular wave, the square wave, the saw tooth wave, the cosine expansion of $\sin x$ and an interrupted square wave. The Gibbs phenomenon and other non-uniform convergence are quite apparent.

VIBRATING STRING. This is an animated set of displays which shows how two travelling waves generate a stationary wave.

PDE1, PDE2, and PDE3 (3 disks). This shows the exact solution of a wave equation for three different initial conditions (PDE1 a smooth hump, PDE2 a step function, PDE3 a triangular hump). Various numerical approximations are then superimposed on the exact solution so that the accuracy and stability of the numerical scheme can be visualized.

TOOLKITS

FINDPOLY - gives student any information about a selected polynomial (graph, derivative, zeros, value) - except what it is. The student has to identify the polynomial. Designed for Calculus I.

TWIDDLE - plots $f(x)$ containing parameters a, b, and c. Then these parameters can be "twiddled" and the graph redrawn. A data set can be fitted in this way. Contains ideas for projects.

FORTUNE - a graphics package that plots $f(x)$ and $g(x)$ containing parameters a, b, and c. Then these parameters can be "tuned" and the graphs redrawn. Contains ideas for projects.

INTEGRAL - performs numerical integration of $f(x)$, by various techniques, (left endpoint, right endpoint, mid-point, trapezoidal, Simpson, Romberg, Gaussian, and Monte Carlo).

ROOTFIND - computes the real roots of $f(x)$, by various techniques, (bisection, Newton, secant, false position). Also plots the graph of $f(x)$ so a root can be found "by eye".

LIMITS - evaluates limits (from above, below, two sided) of $f(x)$ as x goes to a finite or infinite value.

TAYLOR POLYNOMIALS (TOOLKIT) - plots the first 20 polynomials of the Taylor series of the function $f(x)$ about $x = 0$. The user supplies the coefficients explicitly.

SEQUENCES - displays the values of the sequence $a(n)$ and its partial sums, both numerically and graphically. Contains a demo of the proof of Riemann's rearrangement theorem.

INTERPOL - a data set can be entered and then various interpolations (including Lagrange interpolation and cubic splines) are graphically superimposed on this set.

ITERATE - produces iterates of $f(x)$ containing parameters a and b. Gives numerical as well as graphical output. Does orbit diagrams and graphical analysis. Contains ideas for projects.

VENN DIAGRAMS - graphically displays unions, intersections, complements, etc. of user entered set expressions. Also has game feature to test the user's understanding of these concepts.

COMPLEX NUMBERS - graphically displays the sum, product, quotient, conjugate, etc. of complex numbers. Also has game feature to test the user's understanding of these concepts.

LINALG - a comprehensive linear algebra package that does matrix and vector manipulations. Contains ideas for projects.

FOURIER SERIES (TOOLKIT) - plots the first 20 Fourier "polynomials" of the Fourier series of $f(x)$. Will compute the Fourier coefficients numerically, or the user can supply them explicitly.

SLOPES - plots direction fields and integral curves (by Euler's method or Runge Kutta 4) for first order differential equations, $dy/dx = F(x, y)/G(x, y)$.

OLDES - plots numerical solutions of 1st and 2nd order ordinary linear differential equations containing parameters a, b, and c. Then these parameters can be "tuned" and the solution redrawn. A user supplied function, as well as a user supplied power series, can also be plotted.

DIVISION ALGORITHM - user enters real polynomials with integer coefficients. This package will help the user find GCDs, as well as perform Sturm's algorithm, and solve Bezout's identity. The coefficients of all polynomials are stored and shown as rational numbers.

IMPLICIT FUNCTIONS - plots implicit functions defined by $f(x,y) = 0$, as well as contour lines for surfaces $z = f(x,y)$.

POLAR EQUATIONS - graphs polar equations and two-dimensional parametric equations containing parameters a, b, and c. Then these parameters can be "tuned" and the graphs redrawn.

CONVERT UNITS - a numerical package for converting units from one system to another.

Harvey, John, Demana, Franklin, and Waits, Bert K. **"Proceedings of the Third International Conference on Technology in Collegiate Mathematics"** Addison-Wesley Publishing Company, Inc. 1992