Bifurcation of Asymptotically Periodic Solutions of Volterra Integral Equations

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A multiparameter bifurcation theory is developed for families of asymptotically periodic solutions of a general Volterra integal system. A Hopf-type bifurcation theorem follows as a corollary. Within the bifurcating family is a strictly periodic solution to which all other solutions in the family are asymptotically convergent. A characterization of the stable manifold of the bifurcating periodic solution is thus obtained as an existence result. An application is made to an integral equation arising in population dynamics.

1. Introduction

Our concern in this paper is with the existence of asymptotically (nontrivial) periodic solutions of the general system of Volterra equations

$$x(t)=g(t)+\int_0^t K(s)x(t-s)\,ds+r(x), \tag{V}$$

where r is a general operator about which more is assumed below, $|K(s)|\in L^1[0,+\infty)$, and $g(t)\to 0$ as $t\to+\infty$. The approach taken here utilizes a multiparameter (Liapunov–Schmidt-type) bifurcation theory, where K and r depend on $m\geqslant 1$ real parameters, the main result of which is Theorem 5.1. A one-parameter Hopf-type bifurcation theorem (Theorem 5.2) follows as a corollary. These theorems describe the existence of a bifurcating family of asymptotically period solutions. More explicitly it is shown that for certain parameter values (near certain critical values) there is a family of forcing functions $g(t)$ each of which gives rise to an asymptotically periodic solution of (V). Furthermore, there is a unique $g(t)$ in each of these families for which the solution of (V) is strictly periodic and all other forcing functions in the family give rise to solutions that are asymptotically convergent to this periodic one. Also, if the "limiting equation" [namely, (V_p) in Sec. 2 below] of (V) is autonomous in the sense that translates of solutions are still solutions, then the families of forcing

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functions whose solutions are asymptotic to translates of the periodic solution can be unioned to obtain the family of forcing functions whose solutions are asymptotic to some translate of the periodic solution.

Thus, one feature of the approach taken here is that the nature of the stability of the nontrivial periodic solution is obtained as the result of the existence theorem (rather than from a separate stability analysis, such as a Floquet theory).

The analysis is carried out on a direct sum space consisting of a (fixed period) space of periodic functions added to a space of asymptotically zero functions. Equation (V) is then decomposed by projection into a "limiting equation" to be solved on the periodic space, whose bifurcation theory is treated in Sec. 3 (as a generalization of the results in Cushing [4]), and an equation on the space of asymptotically zero functions, which is treated in Sec. 4. The main results are put together in Sec. 5 and an example from the theory of population dynamics is worked out in Sec. 6.

2. Preliminaries

Let BC_0 be the Banach space of n-vector valued functions $x(t)$, continuous for $t \geqslant 0$, that satisfy $x(t) \to 0$ as $t \to +\infty$. Let $C(p)$ be the Banach space of continuous p-periodic functions. Both of these spaces have the supremum norm $|\cdot|_\infty$. Set $A(p) = C(p) \oplus BC_0$, a space whose elements are asymptotically p-periodic.

The following hypothesis will be made on the operator $r(x)$ in (V).

H1. $r: A(p) \to A(p)$ is continuous for some $p > 0$, in which case we can write $r(x) = r_p(x) + r_0(x)$, $r_p: A(p) \to C(p)$, and $r_0: A(p) \to BC_0$. Assume $r_p(0) = 0$ and $r(x_p + x_0) - r(x_p) \in BC_0$ for all $x_p \in C(p)$ and $x_0 \in BC_0$.

In our analysis below we will use the following equivalent hypothesis.

H1′. $r: A(p) \to A(p)$ is continuous for some $p > 0$ such that $r_p(0) = 0$ and $r_p(x_p + x_0)$ is independent of $x_0 \in BC_0$ for every $x_p \in C(p)$ [i.e., $r_p(x_p + x_0) = r_p(x_p)$].

Lemma 1. *Hypotheses* H1 *and* H1′ *are equivalent.*

PROOF. If H1′ holds, then

$$r(x_p + x_0) = r_p(x_p) + r_0(x_p + x_0).$$

Subtracting $r(x_p) = r_p(x_p) + r_0(x_p)$ from $r(x_p + x_0)$, we find that

$$r(x_p + x_0) - r(x_p) = r_0(x_p + x_0) - r_0(x_p) \in BC_0,$$

so that H1 holds.

Conversely, if H1 holds, then

$$r(x_p+x_0)-r(x_p)=r_p(x_p+x_0)+r_0(x_p+x_0)$$
$$-r_p(x_p)-r_0(x_p)\in BC_0,$$

which implies $r_p(x_p+x_0)-r_p(x_p)=0$, i.e., H1′ holds. □

Note that hypotheses H1 and H1′ both imply that r: $BC_0 \to BC_0$ (just let $x_p=0$ in H1′).

The following hypothesis will be made on the kernel K.

H2. $K(t)$ is an $n\times n$ matrix valued function of $t\geqslant 0$ for which $|K(t)|\in L^1[0,+\infty)$.

Lemma 2. *If $K(t)$ satisfies* H2, *then for all $x\in A(p)$ the linear operator $Lx:=\int_0^t K(s)x(t-s)\,ds$ can be written as $Lx=L_px+L_0x$, where*

$$L_px:=\int_0^\infty K(x)x_p(t-s)\,ds\in C(p),$$

$$L_0x:=-\int_t^\infty K(x)x_p(t-s)\,ds+\int_0^t K(t-s)x_0(s)\,ds\in BC_0.$$

PROOF. That $Lx=L_px+L_0x$ and $L_p\in C(p)$ is obvious. The bound

$$\left|\int_t^\infty K(s)x_p(t-s)\,ds\right|\leqslant |x_p|_\infty\int_t^\infty |K(s)|\,ds$$

together with H2 shows that the first integral in L_0x lies in BC_0. To show that the second integral in L_0x also lies in BC_0 let $\varepsilon>0$ be arbitrary and choose $T=T(\varepsilon)>0$ such that $|x_0(t)|<\varepsilon/\int_0^\infty|K(s)|\,ds$ for $t\geqslant T$. Then for $t\geqslant T$

$$\left|\int_0^t K(s)x_0(t-s)\,ds\right|\leqslant\int_0^T|K(t-s)|\,ds\,|x_0|_\infty+\varepsilon,$$

which implies that $\limsup_{t\to+\infty}|\int_0^t K(s)x_0(t-s)\,ds|\leqslant\varepsilon$. Since $\varepsilon>0$ is arbitrary we see that $\lim_{t\to+\infty}\int_0^t K(s)x_0(t-s)\,ds=0$. □

Under hypotheses H1 and H2, Eq. (V), as an equation for $x=x_p+x_0\in A(p)$, can be decomposed into the equivalent, uncoupled pair of systems

$$x_p(t)=\int_0^\infty K(s)x_p(t-s)\,ds+r_p(x_p),\qquad x_p\in C(p),\tag{V_p}$$

$$x_0(t)=g(t)-\int_t^\infty K(s)x_p(t-s)\,ds+r_0(x_p+x_0)$$
$$+\int_0^t K(s)x_0(t-s)\,ds.\tag{V_0}$$

The motivation for hypothesis H1 (or H1′) is both the decoupling of (V_p) and (V_0) and the application to "higher order" operators $r(x)$ such as

$$r(x)=\int_0^t K(s)x^2(t-s)\,ds,$$

which can be seen to satisfy H1′ by noting that

$$r(x_p+x_0)=\int_0^\infty K(s)x_p^2(t-s)\,ds-\int_t^\infty K(s)x_p^2(t-s)\,ds$$
$$+\int_0^t K(s)\big(2x_p(t-s)x_0(t-s)+x_0^2(t-s)\big)\,ds,$$

from which it is easy to see that

$$r_p(x_p+x_0)=\int_0^\infty K(s)x_p^2(t-s)\,ds,$$

and hence that H1′ holds.

Or more generally, consider the operator

$$r(x)=\int_0^t K(t-s)f(x(s))\,ds,$$

where K satisfies H2 and $f\colon R^n\to R^n$, $f(0)=0$, is continuous. For $x=x_p+x_0\in A(p)$ we can write $r(x)=r_p(x)+r_0(x)$, where

$$r_p(x)=\int_0^\infty K(s)f\big(x_p(t-s)\big)\,ds,$$

$$r_0(x)=-\int_t^\infty K(s)f\big(x_p(t-s)\big)\,ds$$
$$+\int_0^t K(s)\big(f\big(x_p(t-s)+x_0(t-s)\big)-f\big(x_p(t-s)\big)\big)\,ds.$$

Clearly, $r_p(x)\in C(p)$ and is independent of x_0. As in the proof of Lemma 2, $r_0(x)\in BC_0$. Thus, $r\colon A(p)\to A(p)$ for any p and satisfies H1.

3. Bifurcation of Periodic Solutions of (V_p)

The purpose of this section is to prove the existence of nontrivial periodic solutions of equations of the from (V_p). This will be done by means of a multiparameter bifurcation theorem that is a generalization of that in Cushing [4]. As a corollary we shall obtain a classical one-parameter, Hopf-type bifurcation theorem for (V_p).

Our approach is often referred to as the "Liapunov–Schmidt method"; specifically, we shall utilize Theorem 1 of Cushing [3] (also see Ref. [4]). The key requirement for this method is the establishment of a Fredholm alternative for the linear systems

$$Ly := y(t) - \int_0^\infty K(s) y(t-s)\,ds = 0, \tag{H}$$

$$Lw := w(t) - \int_0^\infty K(s) w(t-s)\,ds = f(t), \tag{NH}$$

where $f \in C(p)$ for some fixed period $p > 0$ and K satisfies H2. Associated with (H) is the adjoint system

$$z(t) - \int_0^\infty K^{t}(s) z(t+s)\,ds = 0, \tag{A}$$

where the superscript t denotes the complex conjugate transpose.

For complex n vectors $v = \mathrm{col}(v_i)$, $w = \mathrm{col}(w_j)$, let $(v, w) := \Sigma_i v_i w_i^*$, where $*$ denotes complex conjugation. Let $L^2(p)$ denote n-vector valued, p-periodic functions that are square summable on $0 \leqslant t \leqslant p$ and denote $(f, g) = p^{-1}\int_0^p (f(t), g(t))\,dt$. By a solution of (NH), (H), or (A) in $L^2(p)$ we mean functions that satisfy these equations almost everywhere.

Lemma 3. (a) *The homogeneous system* (H) *can have at most a finite number m of independent p-periodic solutions in $L^2(p)$. The adjoint system* (A) *also has exactly m independent p-periodic solutions in $L^2(p)$.*

(b) *If $m = 0$, then* (NH) *has, for each $f \in L^2(p)$, a unique solution $w \in L^2(p)$.*

(c) *If $m \geqslant 1$ and $z_i \in L^2(p)$, $1 \leqslant i \leqslant m$, are independent p-periodic solutions of* (A), *then* (NH) *has a solution $w \in L^2(p)$ if and only if $(z_i, f) = 0$ for all $1 \leqslant i \leqslant m$.*

PROOF. (a) There is a one–one correspondence between functions in $L^2(p)$ and square summable sequences of complex vectors (Fourier coefficients) a_n, $n \geqslant 0$. The function

$$y(t) = \sum_{n=-\infty}^{+\infty} a_n \exp(in\omega t), \qquad \omega = 2\pi/p, \qquad a_{-n} = a_n^*,$$

solves (H) if and only if the a_n solve

$$(I - K_n) a_n = 0, \qquad n \geqslant 0, \tag{3.1}$$

where $K_n := \int_0^\infty K(s) \exp(-in\omega s)\,ds$. By the Riemann–Lebesgue theorem [6], $K_n \to 0$ as $n \to +\infty$, which implies $\det(I - K_n) \to 1$ as $n \to +\infty$. Thus for all sufficiently large n, $\det(I - K_n) \neq 0$ and $a_n = 0$. This implies (H) has at most a finite number of independent p-periodic solutions in $L^2(p)$. That (A) also has m independent p-periodic solutions follows from the fact that the coefficients of a solution $z(t) = \Sigma_{n=-\infty}^{+\infty} b_n \exp(in\omega t)$ must solve the adjoint system of (3.1):

$$(I-K_n)^t b_n=0. \tag{3.2}$$

(b) $x(t)=\Sigma_{n=-\infty}^{+\infty} a_n \exp(in\omega t)$ solves (NH) if and only if

$$(I-K_n)a_n=f_n, \tag{3.3}$$

where the f_n are the Fourier coefficients of $f(t)$. If $m=0$, then (3.1) has no nontrivial solutions for all $n \geq 0$, which implies (3.2) has a unique solution $a_n=(I-K_n)^{-1}f_n$ for all $n \geq 0$. That these Fourier coefficients define a function in $L^2(p)$ follows from the facts that all K_n are bounded by $\int_0^\infty |K(s)|\, ds$ and $\det(I-K_n)\to 1$ (and is consequently bounded away from zero), which imply that $|a_n| \leq c|f_n|$ for some constant $c>0$ independent of n. Then, since $f\in L^2(p)$ implies the f_n are square summable, it follows from this bound that the a_n are square summable.

(c) This statement follows from the necessary and sufficient orthogonality conditions for the solvability of (3.3) when (3.1) and (3.2) have nontrivial solutions. That the sequence of Fourier coefficients still defines a function in $L^2(p)$ follows from the bound $|a_n| \leq c|f_n|$, which is still valid for sufficiently large n and hence implies a_n is square summable. □

The above Fredholm alternative for (NH) is stated on $L^2(p)$. Under the following additional hypothesis on K it is valid on the space $C(p)$:

H3. $K(s)$ satisfies H2 and $(1+s)^\gamma |K(s)|\in L^2[0,+\infty)$ for some $\gamma>1/2$.

Corollary 3.1. *If $K(s)$ satisfies* H3, *then Lemma* 3 *holds with $L^2(p)$ replaced by $C(p)$.*

PROOF. It is only necessary to show that the solutions constructed in the proof of Lemma 3 lie in $C(p)$ when f lies in $C(p)$.

First we show that these solutions are bounded. Write

$$\int_0^\infty K(s)w(t-s)\,ds=\int_0^p K(s)w(t-s)\,ds+\int_p^\infty K(s)w(t-s)\,ds.$$

The Cauchy–Schwarz inequality implies that the first integral satisfies the bound

$$\left|\int_0^p K(s)w(t-s)\,ds\right| \leq |K|_2|w|_2$$

for $t\geq 0$ $[|w|_2^2:=\int_0^p |w(s)|^2\,ds]$ and hence is bounded in $t\geq 0$. Also, we have for $t \geqslant 0$

$$\left|\int_p^{\infty} K(s)w(t-s)\,ds\right|$$

$$\leqslant \left(\int_p^{\infty}(1+s)^{2\gamma}|K(s)|^2\,ds\right)^{1/2}\left(\int_p^{\infty}(1+s)^{-2\gamma}|w(t-s)|^2\,ds\right)^{1/2}$$

$$\leqslant \left(\int_0^{\infty}(1+s)^{2\gamma}|K(s)|^2\,ds\right)^{1/2}\left(\sum_{j=1}^{\infty}\int_{jp}^{(j+1)p}s^{-2\gamma}|w(t-s)|^2\,ds\right)^{1/2}$$

$$\leqslant \left(\int_0^{\infty}(1+s)^{2\gamma}|K(s)|^2\,ds\right)^{1/2} p^{-\gamma}|w|_2\left(\sum_{j=1}^{\infty} j^{-2\gamma}\right)^{1/2}.$$

Thus H3 implies that $\int_0^{\infty}K(s)w(t-s)\,ds$ is bounded in $t \geqslant 0$. It follows that, since $f \in C(p)$ is bounded in $t \geqslant 0$, Eq. (NH) implies that $w(t)$ is bounded in $t \geqslant 0$.

Finally, we argue that the solution $w(t)$ is continuous by showing that the integral in (NH) is continuous in $t \geqslant 0$ when $w \in L^2(p)$ is bounded. But this follows from H3 and the identity

$$\int_0^{\infty}K(s)w(t+\varepsilon-s)\,ds-\int_0^{\infty}K(s)w(t-s)\,ds$$

$$=\int_0^{\infty}(K(s+\varepsilon)-K(s))w(t-s)\,ds+\int_0^{\varepsilon}K(s)w(t+\varepsilon-s)\,ds.$$

The first integral tends to zero as $\varepsilon \to 0$ [6, p. 199]. The last integral tends to zero as $\varepsilon \to 0$ because w is bounded and $|K| \in L^1[0,+\infty)$. □

Corollary 3.1 is a generalization of the Fredholm alternative given in Cushing [4].

Consider now the equation

$$Lx := x(t)-\int_0^{\infty}K(s)x(t-s)\,ds = r(x,\lambda), \qquad \lambda \in R^m, \qquad (V_p^{\lambda})$$

where K satisfies H3 and r, which now depends on m real parameters $\lambda = \operatorname{col}(\lambda_j)$, satisfies H4 below. Let $N(L)$ denote the null space in $C(p)$ of the linear operator L defined in (H).

H4. $r: C(p)\times R^m \to C(p)$ for some p, where $r(\varepsilon x,\lambda)=\varepsilon\bar{r}(x,\lambda,\varepsilon)$ for all small ε and all $(x,\lambda)\in C(p)\times R^m$, where $\bar{r}: C(p)\times R^m\times R \to C(p)$ is $q \geqslant 1$ times continuously (Fréchet) differentiable and satisfies $\bar{r}(y,0,0) = \bar{r}_x(y,0,0)=0$ for some $y \in N(L)$.

Corollary 3.1 implies that the range of the bounded linear operator $L: C(p) \to C(p)$ has finite codimension and that both $N(L)$ and the range of L are closed and admit bounded projections. These facts and H4 together with Theorem 1 of Cushing [3] immediately yield the following general bifurcation theorem for (V_p^λ).

Theorem 3.1. *Assume* $K(s)$ *satisfies* H3 *and that* r *satisfies* H4. *Assume that the linear system* (H) *has, for period* p *as in* H4, *exactly* $m \geq 1$ *independent* p*-periodic solutions and let* $z_i(t) \in C(p)$, $1 \leq i \leq m$, *denote* m *independent* p*-periodic solutions of the adjoint system* (A). *If*

H5. $d := \det(z_i, \bar{r}_{\lambda_j}(y, 0, 0)) \neq 0$,

then (V_p^λ) *has* p*-periodic solutions of the form* $x(t) = \varepsilon y(t) + \varepsilon z(t, \varepsilon)$ *with* $\lambda = \lambda(\varepsilon)$ *for all small* ε, *where* y *is as in* H4, $z(\cdot, \varepsilon) \in C(p)$, $z(t, 0) \equiv 0$, $\lambda(0) = 0$, *and* z, λ *are* q *times continuously* (*Fréchet*) *differentiable in* ε.

REMARKS

1. In most applications $\lambda \in R^m$ is the difference between a system parameter and a critical value of this parameter (at which the linearization has nontrivial periodic solutions) and x is the difference between the dependent variable and some equilibrium state.
2. In Theorem 3.1 use is made of m explicitly appearing system parameters and a branch of periodic solutions of a *fixed* period p is found. In applications the parameters in the vector λ need not all be explicitly appearing in at least the original form of the system, but can be introduced into the analysis by means of rescaling of independent variables. For example, in one-parameter (Hopf-type) bifurcation theorems for autonomous systems only one explicitly appearing parameter is used while $m = 2$. However, one can obtain such Hopf-type bifurcation theorems by changing from variable t to t/p (thereby introducing a second parameter p, the unknown period, into the analysis) and applying Theorem 3.1 on the space of 1-periodic functions (or any other fixed period space). This is in fact often the approach taken in proofs of Hopf-type theorems [4, 8, 9, 10], and we shall apply it to obtain such a theorem for (V_p^λ) in Corollary 2 below. For a further, more detailed discussion of the connection between Theorem 3.1 and Hopf-type bifurcation theorems (at least for integrodifferential equations, but the discussion carries over to integral equations unchanged) see Cushing [2, 3].
3. System (V_p^λ) need not be autonomous in the sense that r, and hence the system, may determine the period p in H3 and Theorem 3.1. Thus, Theorem 3.1 can apply to nonautonomous, but periodic systems. For

applications to such a case and for further generalization of Theorem 1 in Cushing [3] used above, see Ref. [5].

4. Since Theorem 3.1 is only a local result, it is only necessary that H4 hold on some neighborhood of $(x, \lambda)=(0,0)$, $\varepsilon=0$. We have stated H4 as above simply for simplicity and clarity.

Next we derive a more classical, Hopf-type bifurcation theorem for (V_p) using Theorem 3.1 and the approach mentioned in Remark 2. The only complication in proving this corollary arises in showing that the nondegeneracy condition H4 is equivalent to the familiar transversal crossing of "eigenvalues" across the imaginary axis. This was done for the scalar ($n=1$) case in Cushing [4].

Consider the system

$$x(t)-\int_0^{\infty} K(s,\mu)x(t-s)\,ds=h(x,\mu), \qquad \mu\in R^1, \qquad (V_p^{\mu})$$

where K and h now depend on one real parameter μ and where h is "higher order" in x. More specifically,

H6. $h\colon C(p)\times R\to C(p)$ for all p, and if $P\colon C(2\pi)\to C(p)$ is the operator defined by $Px=x(2\pi t/p)$, then the composite operator $h(Px,\mu)$ can be written $h(Px,\mu)=h_1(x,p,\mu)\colon C(2\pi)\times R^2\to C(2\pi)$, where h_1 satisfies $h_1(\varepsilon x, p, \mu)=\varepsilon^2\bar{h}_1(x,p,\mu)$ for small ε and all $(x,p,\mu)\in C(2\pi)\times R^2$, where $\bar{h}$ is $q\geqslant 1$ times continuously (Fréchet) differentiable.

Clearly, if h satisfies H6, then h_1 satisfies H4. In order to carry out the change of variables as described in Remark 2 we also assume the following further assumption on the kernel K.

H7. $K(s,\mu)$ is a twice continuously differentiable (with respect to (s,μ)), $n\times n$ matrix valued function that satisfies $|K(s,\mu)|$, $|\partial K(s,\mu)/\partial\mu|\in L^1[0,+\infty)$, $(1+s)^{\gamma}|K(s,\mu)|\in L^2[0,+\infty)$ in s for some $\gamma>1/2$, and $s|K(s,\mu)|\to 0$ as $s\to+\infty$ for all (small) reals μ.

Let $\hat{K}(z,\mu)$ denote the Laplace transform of $K(s,\mu)$. Assume

H8. There exists a continuously differentiable (complex) root $z=z(\mu)$ of the characteristic equation

$$\Delta(z,\mu):=\det(I-\hat{K}(z,\mu))=0$$

such that

a. $z(0)=i\omega_0$, $\omega_0>0$, and $\operatorname{Re} z'(0)>0$,

b. $\Delta_z(i\omega_0,0)\neq 0$, and

c. one is an algebracally simple (i.e., unrepeated) eigenvalue of $K(i\omega_0,0)$.

Theorem 3.2. *Under hypotheses* H6, H7, *and* H8, *Eq.* (V_p^μ) *has nontrivial periodic solutions of the form*

$$x(t)=\varepsilon y(2\pi t/p)+\varepsilon z(2\pi t/p,\varepsilon),\qquad \mu=\mu(\varepsilon),$$
$$p=p_0+p(\varepsilon),\qquad p_0=2\pi/\omega_0,$$

for small ε, *where* $\mu(\varepsilon), p(\varepsilon)$ *are* q *times continuously differentiable real valued functions for which* $\mu(0)=p(0)=0$ *and where* $y(\cdot), z(\cdot,\varepsilon)$ *are* 2π-*periodic functions,* $z(t,0)\equiv 0$.

PROOF. Changing variables from t to $2\pi t/p$ in (V_p^μ), we arrive at a system of the form

$$x(t)-(\lambda_1+\omega_0^{-1})\int_0^\infty K((\lambda_1+\omega_0^{-1})s,\lambda_2)x(t-s)\,ds=h_1(x,\lambda_1,\lambda_2),$$

where $\lambda_1=\omega^{-1}-\omega_0^{-1}$, $\lambda_2=\mu$, $\omega=2\pi/p$, which by H7 can be written

$$x(t)-\omega_0^{-1}\int_0^\infty K(\omega_0^{-1}s,0)x(t-s)\,ds$$
$$=r(x,\lambda_1,\lambda_2),$$
$$r:=\left[\int_0^\infty\left(K(\omega_0^{-1}s,0)+\omega_0^{-1}sK_s(\omega_0^{-1}s,0)\right)x(t-s)\,ds\right]\lambda_1$$
$$+\left[\omega_0^{-1}\int_0^\infty K_\mu(\omega_0^{-1}s,0)x(t-s)\,ds\right]\lambda_2+\rho(x,\lambda_1,\lambda_2),$$

where ρ (hence r) satisfies H4 and $\bar\rho_{\lambda_j}(x,0,0)=0$. Theorem 3.1 now applies with $m=2$ to this equation provided the nondegeneracy conditions H5 holds.

The number d in H5 is, for the above equation, a 2×2 determinant in which

$$\bar r_{\lambda_1}(y,0,0)=\int_0^\infty\left(K(\omega_0^{-1}s,0)+\omega_0^{-1}sK_s(\omega_0^{-1}s,0)\right)y(t-s)\,ds$$
$$=\int_0^\infty\frac{d}{ds}\left(sK(\omega_0^{-1}s,0)\right)y(t-s)\,ds$$
$$\bar r_{\lambda_2}(y,0,0)=\omega_0^{-1}\int_0^\infty K_\mu(\omega_0^{-1}s,0)y(t-s)\,ds,$$

where $y=\operatorname{Re} v\exp(it)$ is a 2π-periodic solution of

$$y(t)-\omega_0^{-1}\int_0^\infty K(\omega_0^{-1}s,0)y(t-s)\,ds=0,$$

that is, $v\neq 0$ is an n vector satisfying

$$(I-\hat K(i\omega_0,0))v=0.$$

Note that an integration by parts yields

$$\bar{r}_{\lambda_1}(y,0,0)=\int_0^\infty sK(\omega_0^{-1}s,0)y'(t-s)\,ds.$$

The adjoint solutions $z_i(t)$ appearing in H5 are $z_1(t)=\operatorname{Re} w\exp(it)$ and $z_2(t)=\operatorname{Im} w\exp(it)$, where $w\neq 0$ is an n vector satisfying

$$\left(I-\hat{K}(i\omega_0,0)\right)^{t}w=0.$$

It is easy to see (from Jordan form) that H8(c) implies that we may choose v and w such that $(v,w)=1$.

A lengthy but straightforward calculation shows that if $v=a+ib$, $w=c+id$, and if we define the matrices

$$C_1:=\int_0^\infty sK(\omega_0^{-1}s,0)\cos s\,ds, \qquad S_1:=\int_0^\infty sK(\omega_0^{-1}s,0)\sin s\,ds,$$

$$C_2:=\int_0^\infty \omega_0^{-1}K_\mu(\omega_0^{-1}s,0)\cos s\,ds, \quad S_2:=\int_0^\infty \omega_0^{-1}K_\mu(\omega_0^{-1}s,0)\sin s\,ds,$$

then

$$\begin{aligned}4d=&\left[(C_1a,d)-(C_1b,c)+(S_1b,d)+(S_1a,c)\right]\\ &\cdot\left[(C_2a,d)+(S_2b,d)+(S_2a,c)-(C_2b,c)\right]\\ &-\left[(C_2a,c)+(S_2b,c)-(S_2a,d)+(C_2b,d)\right]\\ &\cdot\left[(S_1a,d)-(S_1b,c)-(C_1a,c)-(C_1b,d)\right].\end{aligned}$$

We have to show that this constant d is nonzero under H8. This will be done by arguing that d is a nonzero multiple of $\operatorname{Re} z'(0)$.

Let $v(\mu), w(\mu)$ be continuously differentiable n-vectored values functions of μ satisfying

$$(I-K(z(\mu),\mu))v(\mu)=0, \qquad (I-\hat{K}(\zeta(\mu),\mu))^{t}w(\mu)=0,$$

where $z(\mu), \zeta(\mu)$ are roots of $\Delta(z,\mu)=0$, $\det(I-\hat{K}(\zeta,\mu))^{t}=0$ satisfying $z(0)=i\omega_0$, $\zeta(0)=i\omega_0$, respectively. Then $v(0)=v$, $w(0)=w$. Since $(v(0),w(0))=1$, it follows that $(v(\mu),w(\mu))\neq 0$ for small μ, and hence without loss in generality we can assume $(v(\mu),w(\mu))=1$ for all small μ. Thus

$$(v'(0),w(0))+(v(0),w'(0))=0 \tag{3.4}$$

and a differentiation of $(\hat{K}(z(\mu),\mu)v(\mu),w(\mu))=(v(\mu),w(\mu))=1$ with respect to μ yields

$$\begin{aligned}&\left(\hat{K}(i\omega_0,0)v(0),w'(0)\right)+\left(\hat{K}(i\omega_0,0)v'(0),w(0)\right)\\ &\quad+\left(\hat{K}_z(iw_0,0)z'(0)v(0)+\hat{K}_\mu(iw_0,0)v(0),w(0)\right)=0.\end{aligned}$$

But since $\hat{K}(iw_0,0)v(0)=v(0)$, $\hat{K}(i\omega,0)^{t}w(0)=w(0)$, we find from (3.4) that the first two terms of this equation add to zero, so that we obtain

$$\left(\hat{K}_z(i\omega_0,0)v,w\right)z'(0)=-\left(\hat{K}_\mu(iw_0,0)v,w\right). \tag{3.5}$$

Using

$$\hat{K}_z(iw_0,0)=-\omega^{-2}(C_1-iS_1) \quad \text{and} \quad \hat{K}_\mu(iw_0,0)=\omega_0^{-1}(C_2-iS_2),$$

multiplying both sides by the complex conjugate of $(\hat{K}_z(i\omega_0,0)v,w)$, and taking the real part of the result, we obtain from (3.5) the equation

$$4|\left(\hat{K}_z(i\omega_0,0)v,w\right)|^2\operatorname{Re}z'(0)=\omega_0 d. \tag{3.6}$$

Hypothesis H8(c) and the results of the appendix show that $(\hat{K}_z(i\omega_0,0)v,w)$ is a nonzero multiple of $\Delta z(i\omega_0,0)$ and hence by H8(b) is nonzero. Identity (3.6) and H8(a) imply that $d\neq 0$. □

Note the similarity between Theorem 3.2 and the classical Hopf bifurcation theorem for ordinary differential equations. Roughly speaking, Theorem 3.2 says that bifurcation occurs when a root of the characteristic equation transversally crosses the imaginary axis (not through the origin) as μ passes through zero [see H8(a)] and that the root on the imaginary axis when $\mu=0$ is "simple" [in the sense of H8(b, c)].

In many applications the kernel has compact support in s. For such cases H8 can be restated so that the conditions described there need only hold on the support of K and Theorem 3.2 remains valid.

4. A Stable Manifold Theorem

Define F to be the set of forcing functions $f\in BC_0$ such that the unique solution $x=Sf$ of the linear Volterra integral system

$$x(t)=f(t)+\int_0^t K(s)x(t-s)\,ds \tag{4.1}$$

lies in BC_0. Clearly, F is a linear subspace of BC_0 and the operator S: $F\to BC_0$ is linear.

Assume now that $x_p\in C(p)$ is a solution of the limiting equation (V_p). For any $f\in F$ define $g\in BC_0$ by

$$g(t):=f(t)+\int_t^\infty K(s)x_p(t-s)\,ds-r_0(x_p+Sf). \tag{4.2}$$

[That $g\in BC_0$ follows from Lemma 2 with $x\in C(p)$, i.e., with $x_0=0$.] Then the solution of the linear system

$$\begin{aligned} x(t)=g(t)-\int_t^\infty K(s)x_p(t-s)\,ds+r_0(x_p+Sf) \\ +\int_0^t K(s)x(t-s)\,ds \end{aligned}$$

is $x=Sf\in BC_0$. Thus, $x_0=x=Sf$ actually solves (V_0) for the function $g(t)$ given by (4.2). Define the operator $G: F\to BC_0$ by

$$G(f):=f(t)+\int_t^{\infty}K(s)x_p(t-s)\,ds-r_0(x_p+Sf)\tag{4.3}$$

and let $M=M(x_p)$ be the range of G.

Theorem 4.1. *Assume r satisfies* H1 *and that K satisfies* H2. *Then* $x\in A(p)$ *is an asymptotically p-periodic solution of* (V) *if and only if* $g\in M(x_p)$, *where* $x_p\in C(p)$ *solves* (V_p) *and* $x_0=Sf$, *where* $g=G(f)$.

PROOF. We have already shown that if $x_p\in C(p)$ solves (V_p) and if $g=G(f)\in M(x_p)$, then $x=x_p+Sf\in A(p)$ solves (V).

Conversely, suppose $x=x_p+x_0\in A(p)$ solves (V). By direct sum decomposition and Lemma 2, x_p solves (V_p) and x_0 solves (V_0). Define

$$f(t):=g(t)-\int_t^{\infty}K(s)x_p(t-s)\,ds+r_0(x_p+x_0)\in BC_0.$$

Then clearly, $x_0\in BC_0$ solves (4.1) for this f, which means that $f\in F$ and $x_0=Sf$. Thus $g=G(f)\in M(x_p)$. □

REMARKS

1. It is possible to take $x_p=0$ in Theorem 4.1, in which case x in that theorem is asymptotic to zero as $t\to+\infty$.
2. If $x_p\in C(p)$ solves (V_p), then x_p also solves (V) for $g(t)=g_p(t)$, where

   $$g_p(t):=\int_t^{\infty}K(s)x_p(t-s)\,ds-r_0(x_p)\in BC_0.$$

   In other words, there is a forcing function $g\in BC_0$ for Eq. (V) that gives rise to a strictly p-periodic solution. Note that $g=G(0)$, so that in fact $g\in M(x_p)$. Clearly, there is no other forcing function in the manifold $M(x_p)$ that gives rise to a strictly periodic solution of (V).
3. Given any solution $x_p\in C(p)$ of (V_p), Theorem 4.1 completely characterizes the manifold of forcing functions $g\in BC_0$ in Eq. (V) whose corresponding solutions $x(t)$ satisfy $x(t)-x_p(t)\to 0$, and in this sense Theorem 4.1 is a stable manifold theorem concerning x_p.
4. If (V_p) is autonomous in the sense that any t translate of a solution is again a solution (as is often the case in applications), then Theorem 4.1 can be applied to any translate $x_p^{\tau}(t):=x_p(t+\tau)$ of x_p. The union $\bigcup_{\tau} M(x_p^{\tau})$ of the disjoint manifolds $M(x_p^{\tau})$ is clearly the set of all forcing functions $g(t)$ that in Eq. (V) yield solutions asymptotic as $t\to+\infty$ to some translate of $x_p(t)$.

This section will close with the study of some conditions under which a neighborhood of $g_p\in M(x_p)$ is homeomorphic to a neighborhood of $O\in F$.

Consider again the linear equation (4.1). Denoting the Laplace transform of a function $x(t)$ by $\hat{x}(z)$, it follows that

$$\hat{x}(z)=(I-\hat{K}(z))^{-1}\hat{f}(z). \tag{4.4}$$

The behavior of the solution of Eq. (4.1) depends on the location of the roots of the characteristic equation

$$\det(I-\hat{K}(z))=0, \tag{4.5}$$

about which we make the following hypothesis.

H9. The characteristic equation (4.5) has a finite number ν of roots z_k satisfying $\operatorname{Re} z \geqslant 0$. Each z_k has finite (algebraic) multiplicity $m_k > 0$, $1 \leqslant k \leqslant \nu$, and satisfies $\operatorname{Re} z_k > 0$.

The roots of (4.5) occur in complex conjugate pairs. The inverse $(I-\hat{K}(s))^{-1}$ has a Laurent series of the form

$$\sum_{j=1}^{m_k} A_{jk}(z-z_k)^{-j}+H_k(z) \tag{4.6}$$

near $z=z_k$, where A_{jk} is an $n \times n$ matrix and $H_k(z)$ is an analytic matrix valued function. Define the polynomial

$$p(z):=\prod_{k=1}^{\nu}(z-z_k)^{m_k}. \tag{4.7}$$

If we let

$$B(z):=(I-\hat{K}(z))/p(z), \tag{4.8}$$

then $B^{-1}(z)=(I-\hat{K}(z))^{-1}p(z)$ is analytic in $\operatorname{Re} z \geqslant 0$ and (4.4) can be written

$$\hat{x}(z)=B^{-1}(z)\hat{f}(z)/p(z). \tag{4.9}$$

Thus $f \in F$ if and only if $\hat{f}$ factors:

$$\hat{f}(z)=p(z)\hat{u}(z). \tag{4.10}$$

Lemma 4. *Suppose $f=\operatorname{col}(f_i) \in F$. There exists a unique $u^i \in BC_0$ that solves the initial value problem*

$$p(D)u^i=f_i, \qquad 1 \leqslant i \leqslant n, \qquad D=d/dt, \tag{4.11a}$$

$$D^{j-1}u^i(0)=0, \qquad 1 \leqslant j \leqslant n. \tag{4.11b}$$

Furthermore $D^{j-1}u^i \in BC_0$ for $1 \leqslant i,j \leqslant n$ and u^i is given by the formula

$$u^i(t)=-\int_t^{\infty} e^{(t-s)A}\operatorname{col}(0,\ldots,0,f_i(s))\,ds, \tag{4.12}$$

where A is the companion matrix of the polynomial $p(z)$. The Laplace transform of $u(t)=\operatorname{col}(u^i(t))$ satisfies (4.10).

PROOF. It is clear that u^i defined by (4.12) solves (4.11a). That (4.11b) is also satisfied follows from the fact that $\hat{f}(z)$ vanishes at the roots of $p(z)$ (including multiplicities). Thus (4.12) defines the unique solution of (4.11). Since all eigenvalues of A (which are the roots of $p(z)$) lie in the right half-plane $\operatorname{Re} z>0$, it follows from $f\in BC_0$ that $D^{j-1}u^i\in BC_0$ for $1\leqslant i, j\leqslant n$. Equation (4.10) follows immediately from (4.11). □

Let

$$q(z):=\prod_{k=1}^{\nu}(z+z_k)^{m_k}$$

and

$$h(t)=q(D)u(t). \tag{4.13}$$

It follows from Lemma 4 that $h(t)$ can be written

$$h(t)=f(t)+(q(D)-p(D))u \tag{4.14}$$

and hence that $h\in BC_0$. Using (4.10), (4.13), and Lemma 3, we can write (4.9) as

$$\hat{x}(z)=B^{-1}(z)\hat{h}(z)/q(z).$$

We now make the definition

$$\hat{K}_-(z):=I-q(z)B(z)=\hat{K}(z)+(p(z)-q(z))B(z), \tag{4.15}$$

where the second equality follows from (4.8). Assume

H10. Equation (4.15) defines the Laplace transform of a function $K_-(t)$ satisfying $|K_-(t)|\in L^1[0,+\infty)$.

This assumption implies that $x(t)$ solves the equation

$$x(t)=h(t)+\int_0^t K_-(t-s)x(s)\,ds. \tag{4.16}$$

The characteristic equation associated with (4.16) is

$$0=\det\left(I-\hat{K}_-(z)\right)=\det q(z)B(z),$$

which, by the construction of q and B, has no zeros satisfying $\operatorname{Re} z\geqslant 0$. The Paley–Wiener theorem then implies that the resolvent $R_-(t)$ of (4.16) lies in $L^1[0,+\infty)$. This proves

Theorem 4.2. *Suppose* H2, H9, *and* H10 *hold and assume* $f \in F$. *Then the solution* $x = Sf$ *of the linear equation* (4.1) *is given by*

$$x(t) = h(t) + \int_0^t R_-(t-s)h(s)\,ds, \tag{4.17}$$

where $h(t)$ *is determined from* $f(t)$ *by* (4.14) *and* (4.12) *and where* $R_-(t)$, *the resolvent of* (4.16), *satisfies* $\int_0^\infty |R_-(t)|\,dt < +\infty$.

Corollary 4.1. *The linear space F is closed under the hypotheses* H2, H9, *and* H10.

PROOF. Suppose $f_n \in F$ and $f_n \to f \in BC_0$. From (4.12) and (4.14) it follows that the corresponding $h_n \in BC_0$ (as described in Theorem 4.2) converge in BC_0 to an $h \in BC_0$. Equation (4.17) and Lebesgue's dominated convergence theorem combine to imply that the solution of (4.1) corresponding to f lies in BC_0. This in turn implies that $f \in F$ and hence that F is closed.

Corollary 4.2. *The solution operator* S: $F \to BC_0$ *of Eq.* (4.1), *under hypotheses* H2, H9, *and* H10, *and the operator* G: $F \to BC_0$ *defined by* (4.3) *are continuous.*

PROOF. Consider $x = Sf$ for $f \in F$. Equation (4.17) yields the bound

$$|Sf|_\infty \leqslant \left(1 + \int_0^\infty |R_-(s)|\,ds\right)|h|_\infty,$$

and a straightforward computation using (4.12) and (4.14) shows that $|D^j u|_\infty \leqslant c_1 |f|_\infty$ and $|h|_\infty \leqslant c_2 |f|_\infty$ for constants $c_1, c_2 > 0$. Hence the norm of the linear operator S is bounded above by

$$\|S\| \leqslant \left(1 + \int_0^\infty |R_-(s)|\,ds\right)c_2.$$

The continuity of G now follows from the continuity of S and r_0. □

We now have the final

Corollary 4.3. *Suppose the kernel* $K(s)$ *satisfies* H2, H9, *and* H10. *Suppose* r_0: $A(p) \to BC_0$ *is continuously Fréchet differentiable in a neighborhood of* $x = 0$ *and* $r_0'(0) = 0$. *Then for each sufficiently small solution* $x_p \in C(p)$ *of* (V_p) *the operator* G *is a homeomorphism from a neighborhood of* $f = 0$ *in* F *to a neighborhood of* $g_p := G(0) = \int_t^\infty K(s)x_p(t-s)\,ds - r_0(x_p)$ *in* BC_0.

PROOF. From (4.3) we see that for $x_p = 0$ the Fréchet derivative $G'(0) = I$. Thus, $G'(0)$ is also invertible for small $|x_p|_\infty$. □

Corollary 4.3 says roughly that the "size" of the stable manifold of a small solution x_p of (V) (for $g=g_p$) is locally, near $g=g_p$, the same as that of F near $f=0$.

5. The Main Results

We are now ready to state our main results, which are obtained by combining the results of Sec. 3 and 4. Suppose that the Volterra equation (V) contains m real parameters

$$x(t)=g(t)+\int_0^t K(s)x(t-s)\,ds+r(x,\lambda),$$

$$\lambda=\mathrm{col}(\lambda_i)\in R^m, \tag{V^λ}$$

where

H11. for each $\lambda\in R^m$ the operator $r(x,\lambda)$ satisfies H1 and $r_p(x,\lambda)$ satisfies H4

and $K(s)$ satisfies H3. Then the bifurcation Theorem 3.1 can be applied to the "limiting" equation

$$x_p(t)=\int_0^\infty K(s)x_p(t-s)\,ds+r_p(x_p,\lambda) \tag{V_p^λ}$$

to obtain solutions $x_p\in C(p)$ for certain $\lambda\in R^m$. For each such solution x_p, Theorem 4.1 can then be applied to obtain a family of forcing functions $g\in BC_0$ whose corresponding solutions of (V^λ) are asymptotic to x_p as $t\to+\infty$. More explicitly we obtain the following result.

Theorem 5.1. *Suppose that the kernel $K(s)$ satisfies* H3 *and that $r(x,\lambda)$ satisfies* H11. *Assume that the linearized homogeneous system* (H) *has exactly $m\geqslant 1$ independent p-periodic solutions. If the nondegeneracy condition* H5 *holds, then for each small ε there exists a family $M=M(\varepsilon)\subseteq BC_0$ of forcing functions such that for every $g\in M(\varepsilon)$, Eq.* (V^λ) *has a solution $x(t)$ satisfying $x(t)-x_p(t)\in BC_0$, where $x_p\in C(p)$ is the p-periodic solution of* (V_p) *with $\lambda=\lambda(\varepsilon)$ as described in Theorem* 3.1.

By Remark 2 of Sec. 4 there is a unique $g\in M(\varepsilon)$ [in fact $g(t)=g_p(t):=\int_t^\infty K(s)x_p(t-s)\,ds-r_0(x_p,\lambda)$] for which the solution $x(t)$ of (V^λ) is equal to $x_p(t)$. Thus, Theorem 5.1 contains a bifurcation result for p-periodic solutions of (V^λ), but it goes beyond this to describe the stable manifold of $x_p(t)$ in terms of the family of forcing functions $M(\varepsilon)$. Corollary 4.3 can also be applied to describe the "size" of this stable manifold.

Similarly the Hopf-type bifurcation Theorem 3.2 can be combined with Theorem 5.1. Consider

$$x(t)=g(t)+\int_0^t K(s,\mu)x(t-s)\,ds+r(x,\mu), \qquad \mu\in R^1, \qquad (\mathrm{V}^\mu)$$

where

H12. for each $\mu\in R^1$ the operator $r(x,\mu)$ satisfies H1 and $r_p(x,\mu)$ satisfies H6.

We then obtain the following one-parameter, Hopf-type bifurcation result for (V^μ).

Theorem 5.2. *Assume that the kernel* $K(s,\mu)$ *satisfies* H7 *and that* $r(x,\mu)$ *satisfies the condition* H12. *If the root conditions* H8 *hold, then for small* ε *there exists a family of forcing functions* $M=M(\varepsilon)\subseteq BC_0$ *such that for every* $g\in M(\varepsilon)$, *Eq.* (V^μ) *has an asymptotically p-periodic solution* $x(t)$ *satisfying* $x(t)-x_p(t)\in BC_0$, *where* $x_p(t)$ *is the p-periodic solution of* (V_p^μ) *and* $\mu=\mu(\varepsilon)$, $p=p(\varepsilon)$ *as described in Theorem* 3.2.

As in Theorem 5.1 there is a unique $g\in M(\varepsilon)$ whose corresponding solution is strictly $p(\varepsilon)$ periodic and all other solutions for $g\in M(\varepsilon)$ are asymptotic to this periodic solution as $t\to+\infty$. Again, given ε, Corollary 4.3 can be applied to the manifold $M(\varepsilon)$ provided the kernel $K(s,\mu(\varepsilon))$ satisfies H2, H9, and H10.

6. An Application

A nonlinear scalar ($n=1$) Volterra integral equation that appears in a variety of disciplines (e.g., population dynamics, epidemiology, and economics) is the nonlinear renewal equation

$$b(t)=b_0(t)+\int_0^t m(s)f(b(t-s))\,ds. \qquad (6.1)$$

For example, $b(t)$ might be the birth rate of a biological population. In this case, $b_0(t)$ is the (current time t) birth rate of the survivors from the initial population [at $t=0$], $f(b(t-s))$ is the number of individuals born between $t-s$ and $t-s+ds$, and $m(s)$ is the product of the probability of survival to age s times the per unit population birth rate at age s. Thus, the integral in (6.1) is the birth rate at time t due to all individuals born since $t=0$. Here the fecundity function $f(b)$ has been assumed to be dependent on the current birth rate. This fecundity function is typically chosen to be nonnegative, to vanish at or near $b=0$ and decrease or even vanish identically for large b. Also there are typically at least two nonnegative asymptotic equilibria, i.e., constant solutions of the limiting equation

$$b(t)=\int_0^\infty m(s)f(b(t-s))\,ds. \tag{6.2}$$

[Note: $b_0(t)$ is reasonably assumed to be in BC_0 or even to have compact support.] Namely, (6.2) has equilibrium $b(t)\equiv 0$ and is assumed to have a second equilibrium $b(t)\equiv e>0$, where e solves

$$e=f(e)\int_0^\infty m(s)\,ds.$$

Our application will consist in applying our results to the question of the existence on nontrivial, periodic solutions of (6.1) that bifurcate from $e>0$.

Although the results of Sec. 5 are sufficiently general to apply to (6.1) for general fecundity functions f, we choose to make this example more explicit by choosing a specific function f. Namely, we shall consider the equation (see Hoppensteadt [7, p. 11] and Cooke and Yorke [1])

$$b(t)=b_0(t)+\mu\int_0^t\beta(s)b(t-s)[1-b(t-s)]_+\,ds, \tag{6.3}$$

where $\beta(s)\geqslant 0$, $\int_0^\infty\beta(s)\,ds=1$ is normalized, and $\mu>0$ is a constant. Here $\{\xi\}_+=0$ for $\xi\leqslant 0$ and $\{\xi\}_+=\xi$ for $\xi>0$. Thus, in this case fecundity drops to zero when the birth rate b equals zero or unity (which can be accomplished without loss in generality by an appropriate choice of units). Equation (6.3) has equilibria $b(t)\equiv 0$ and $b(t)\equiv e=(\mu-1)/\mu$, so we assume throughout that $\mu>1$ (μ is the expected number of offspring during the life of an individual or, in a sexually reproducing population, of an individual female).

If we let $x(t)=b(t)-e$, then (6.3) becomes an equation for x:

$$x(t)=g(t)+\int_0^t K(s,\mu)x(t-s)\,ds+r(x,\mu),$$

$$K(s,\mu):=(2-\mu)\beta(s),\qquad r(x,\mu):=-\mu\int_0^t\beta(s)x^2(t-s)\,ds,$$

$$g(t):=b_0(t)-(\rho-1)\rho^{-1}\int_t^\infty\beta(s)\,ds.$$

If $\beta(s)$ is twice continuously differentiable and

$$\begin{aligned}&\beta(s)\geqslant 0,\qquad \int_0^\infty\beta(s)\,ds=1,\qquad \int_0^\infty(1+s)^{2\gamma}\beta^2(s)\,ds<\infty\\ &\text{for some } \gamma>\tfrac{1}{2} \text{ and}\\ &s\beta(s)\to 0\quad \text{as } s\to+\infty,\end{aligned} \tag{6.4}$$

then it is not difficult to check that r satisfies H12 and K satisfies H7. Thus in order to obtain Hopf-type bifurcation from Theorem 5.2, we need only fulfill H8. Note that H8(c) automatically holds in the scalar case $n=1$. In H8 the characteristic function for this application is

$$\Delta(z,\mu)=1-(2-\mu)\hat{\beta}(z).$$

Suppose by way of illustration we take

$$\beta(s)=(n/T)^{n+1}(s^n/n!)\exp(-sn/T) \tag{6.5}$$

for some $n=1,2,\ldots$and real $T>0$. Here T is the age of maximum fecundity and n inversely measures the "width" of the "reproductive window" (i.e., the age interval of active reproduction). Such a choice of $\beta(s)$ satisfies all of the conditions in (6.4). The characteristic equation becomes

$$1-(2-\mu)(n/(zT+n))^{n+1}=0, \tag{6.6}$$

which has simple conjugate roots $z=z(\mu)$ that transversally cross the imaginary axis (away from the origin) as required by H8(a, b) if and only if $n\geqslant 2$. In this case the root

$$z(\mu)=n\big((\mu-2)^{1/(n+1)}\exp(i\pi/(n+1))-1\big)/T$$

crosses the axis at $z(\mu_0)=i$, where

$$\mu_0=2+\sec^{n+1}(\pi/(n+1)),$$

$$\omega=(n/T)(\mu_0-2)^{1/(n+1)}\sin(\pi/(n+1)). \tag{6.7}$$

Moreover,

$$\operatorname{Re} z'(\mu_0)=(n/\pi(n+1))(\mu_0-2)^{-n/(n+1)}\cos(\pi/(n+1))\neq 0,$$

so that H8 holds.

We conclude that *a Hopf-type bifurcation of a family of asymptotically periodic solutions of* (6.3) *occurs from the equilibrium $e=(\mu-1)/\mu$ for the kernel* (6.5) *when $n\geqslant 2$ at the critical values μ_0 of μ and for periods near $2\pi/\omega$ given by* (6.7).

Finally, in order to draw the conclusions concerning the stable manifold contained in Corollaries 4.1–4.3, we need to verify H9 and H10. If we restrict μ in such a way that there is exactly one complex conjugate pair of roots of the characteristic equation (6.6) satisfying $\operatorname{Re} z>0$ and that all of the remaining roots satisfy $\operatorname{Re} z<0$, then H9 will hold. This can be done by requiring that $\mu>\mu_0$ when $n=2$, 3, 4, or 5 and that $\mu_0<\mu<2+\sec^{n+1}3\pi/(n+1)$ when $n\geqslant 6$. For such μ hypothesis H9 holds.

The characteristic function in this example can be written

$$\Delta(z,\mu)=\frac{\prod_{j=0}^{n}(z-z_j)}{(z+(n/T))^{n+1}},$$

where

$$z_j = n\left((\mu-2)^{1/(n+1)}\phi_j - 1\right)/T, \qquad j=0,1,\ldots,n,$$

are the roots of the characteristic equation (6.6) and $\phi_j = \exp((2j+1)\pi i/(n+1))$ are the $n+1$ roots of -1. The hypothesis H10 is satisfied if the inverse Laplace transform of

$$\hat{K}_-(z,\rho) := \hat{K}(z,\mu) - \frac{(4z\,\mathrm{Re}\,z_0)\prod_{j=1}^{n-1}(z-z_j)}{(z+(n/T))^{n+1}}$$

lies in $L^1[0,+\infty)$. But this is obvious, since this is certainly true of $\hat{K}(z,\mu)$ and since the second term is a rational function of z whose denominator has degree larger than its numerator and whose poles all lie in the left half-plane (namely at $z=-n/T$).

Appendix

Let $M=M(z)=(m_{ij}(z))$ be an $n\times n$ matrix with differentiable entries $m_{ij}(z)$ and suppose $M(z_0)$ has rank $n-1$ [and hence $\det M(z_0)=0$]. Let $v\neq 0$ and $w\neq 0$ be in the kernel of $M(z_0)$ and its conjugate transpose, respectively. Let $\Delta(z)=\det M(z)$, $\Delta'(z)=d\Delta(z)/dz$, and $M'(z)=(dm_{ij}(z)/dz)$.

Theorem. *$(M'(z_0)v,w)=0$ if and only if $\Delta'(z_0)=0$.*

PROOF. Let $c_{ij}(z)$ be the cofactor of $m_{ij}(z)$ in $M(z)$. Since the rank of $M(z_0)$ is $n-1$, not all $c_{ij}(z_0)=0$, and we assume without loss of generality that $c_{11}(z_0)\neq 0$.

Let $\Delta_i(z)$ denote the determinant of the matrix obtained by differentiating the ith row of $M(z)$. Then as is well known,

$$\Delta'(z) = \sum_{i=1}^{n}\Delta_i(z). \tag{A1}$$

Let $v=\mathrm{col}(v_i)$ and $w=\mathrm{col}(w_i)$. Then

$$\sum_{j=1}^{n} m_{ij}(z_0)v_j = 0, \qquad \sum_{j=1}^{n} m^*_{ji}(z_0)w_j = 0, \qquad 1\leqslant i\leqslant n.$$

Ignoring the kth equation ($1\leqslant k\leqslant n$), we consider the equations

$$\sum_{j=2}^{n} m_{ij}(z_0)v_j = -m_{i1}(z_0)v_1, \qquad \sum_{j=2}^{n} m^*_{ji}(z_0)w_j = -m^*_{1j}(z_0)w_1$$

for $i\neq k$, $1\leqslant i\leqslant n$. Treating these are $n-1$ equations for $v_2,\ldots,v_n$ and $w_2,\ldots,w_n$, respectively, we get, from the usual relation between the determinant, the unknowns, and the Cramer determinants of a linear algebraic system, the identities

$$c_{k1}(z_0)v_j=c_{kj}(z_0)v_1,\qquad c^*_{1k}(z_0)w_j=c^*_{jk}(z_0)w_1 \tag{A2}$$

for $2\leqslant j\leqslant n$, $1\leqslant k\leqslant n$. Since $c_{11}(z_0)\neq 0$ by assumption, these imply (with $k=1$) that $v_1\neq 0$ and $w_1\neq 0$ (for otherwise $v=0$ and $w=0$). Without loss of generality we may assume that v is scaled so that $v_1=c_{11}(z_0)$. Then from (A2) with $k=1$ we find that

$$v_j=c_{1j},\qquad 1\leqslant j\leqslant n. \tag{A3}$$

Clearly,

$$(M'(z_0)v,w)=\sum_{k=1}^{n}\left(\sum_{j=1}^{n}m'_{kj}(z_0)v_j\right)w^*_k. \tag{A4}$$

Case 1. If $c_{k1}(z_0)\neq 0$, then from (A2) and (A3) we obtain

$$c_{ij}=v_j=c_{kj}(z_0)c_{11}(z_0)/c_{k1}(z_0),\qquad 2\leqslant j\leqslant n.$$

Thus

$$\sum_{j=1}^{n}m'_{kj}(z_0)v_j=\Delta_k(z_0)c_{11}(z_0)/c_{k1}(z_0). \tag{A5}$$

Also from (A2) with $k=1$ we have $w^*_j=c_{j1}(z_0)w^*_1/c_{11}(z_0)$ for $2\leqslant j\leqslant n$, from which we see that

$$w^*_1\neq 0 \tag{A6}$$

and $w^*_1=w^*_kc_{11}(z_0)/c_{k1}(z_0)$. Thus

$$\sum_{j=1}^{n}m'_{kj}(z_0)v_jw^*_k+\Delta_k(z_0)w^*_1. \tag{A7}$$

Case 2. If $c_{k1}=0$, then $v_1\neq 0$ and (A2) imply that all $c_{kj}=0$, $1\leqslant j\leqslant n$. This clearly implies that

$$\Delta_k(z_0)=0. \tag{A8}$$

From (A2) we have $c^*_{11}(z_0)w_j=c^*_{j1}(z_0)w_1$ for all $2\leqslant j\leqslant n$. Letting $j=k$, we see that in this case $w_k=0$.

Having considered these two cases we return to (A4) and find that

$$(M'(z_0)v,w)=w^*_1\left(\Delta_1(z_0)+\sum_{k=2}^{n}\sigma_k\Delta_k(z_0)\right),$$

where $\sigma_k = 1$ if $c_{k1} \neq 0$ (Case 1) and $\sigma_k = 0$ if $c_{k1} = 0$ (Case 2). From (A1) we find that because of (A8) in Case 2, it follows that

$$(M'(z_0)v, w) = w_1^* \Delta'(z_0),$$

which proves the theorem. □

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