CHAPTER 1

First Order Equations

In this Chapter we consider first order differential equations of the form

$$x' = f(t, x).$$

A fundamental question concerns the existence of solutions to such an equation. Under what conditions (i.e., for what kind of expressions $f(t, x)$) can we be assured that solutions exist? Another question concerns the number of solutions. We know from calculus that integration problems have infinitely many solutions and, therefore, we anticipate that this is also true for a first order differential equation. On the other hand, in applications there are often requirements (in addition to the differential equation) that serve to select exactly one solution. For a first order differential equation the most common requirement is that the solution $x(t)$ equal a specified value x_0 for a specified value of t, that is to say, that $x(t_0) = x_0$ for a given t_0 and x_0. A fundamental mathematical question is whether the resulting *initial value problem*

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a solution. In this chapter we learn conditions which, when placed on $f(t, x)$, guarantee that this initial value problem has one and only one solution (i.e., has a "unique" solution).

For specialized equations (i.e., for $f(t, x)$ with special properties) one can calculate formulas for solutions. We study some examples in Chapters 2 and 3. However, for most differential equations it is not possible to find solution formulas. Nonetheless, it is possible to obtain useful approximations to solutions of any first order equation, especially with the aid of a computer. In this chapter we study some basic methods for approximating solutions, both graphically and quantitatively. In applications these methods are often sufficient to obtain the desired answers. Other approximation methods appear in Chapter 3.

1. The Fundamental Existence Theorem

We begin with a definition.

Definition 1.1. *A **solution** of a differential equation $x' = f(t, x)$ on an interval $a < t < b$ is a differentiable function $x = x(t)$ that reduces the equation to an identity on the interval, i.e., $x'(t) = f(t, x(t))$ for all values of t from the interval.*[1] *The interval $a < t < b$ may be the whole real line, in which case we say the function is a solution for all t.*

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[1] As a mathematical function $f(t, x)$ has a domain of t and x values. It is assumed, in this definition, that all values of t taken from the interval $a < t < b$ and the corresponding values of $x(t)$ (i.e., the range of the function $x(t)$) lie in the domain of f. Otherwise $f(t, x(t))$ makes no sense.

For the differential equation

$$x' = t^2 \tag{1.1}$$

we have $f(t,x) = t^2$. The function $x(t) = t^3/3 + 1$ is a solution of this equation for all t because $x'(t) = t^2$ equals $f(t, x(t)) = t^2$ for all t.

More generally, the unknown x might appear in $f(t,x)$. For example, for the equation $x' = tx$ we have $f(t,x) = tx$. The function $x(t) = e^{t^2/2}$ is a solution of this equation for all t because $x'(t) = te^{t^2/2}$ and $f(t, x(t)) = tx(t) = te^{t^2/2}$ are identical for all t.

From calculus we know the differential equation (1.1) has infinitely many solutions and the set of all solutions is given by the formula

$$x = \frac{1}{3}t^3 + c \tag{1.2}$$

where c is an arbitrary constant. This is an example of a "general solution" of a differential equation.

Definition 1.2. *The collection of all solutions of the differential equation $x' = f(t,x)$ is called the* ***general solution*** *(or the* ***solution set****).*

An initial condition $x(t_0) = x_0$ selects a particular solution from the general solution. For example, suppose we require that a solution of the equation (1.1) satisfy the initial condition $x(0) = 1$. From the general solution (1.2) we obtain $x(0) = c$ and therefore this initial condition is satisfied by choosing (and only by choosing) $c = 1$. That is to say, there is a unique solution of the initial value problem

$$x' = t^2, \quad x(0) = 1$$

namely, $x = t^3/3 + 1$.

In an initial value problem the "initial" time need not be $t_0 = 0$. For example, we can use the general solution (1.2) to find the unique solution

$$x(t) = \frac{1}{3}t^3 - \frac{11}{3}$$

of the initial value problem

$$x' = t^2, \quad x(2) = -1.$$

In fact, we can solve the general initial value problem

$$x' = t^2, \quad x(t_0) = x_0.$$

using the general solution (1.2) by setting

$$x(t_0) = \frac{1}{3}t_0^3 + c$$

equal to the desired initial value x_0 and solving for

$$c = x_0 - \frac{1}{3}t_0^3.$$

This results in the unique solution

$$x(t) = \frac{1}{3}t^3 + x_0 - \frac{1}{3}t_0^3.$$

EXAMPLE 1.1. *A differential equation for the velocity* $v = v(t)$ *of a falling object subject to the force of gravity and air resistance is* $v' = f(t, v)$ *where* $f(t, v) = g - k_0 v$. *Here* g *and* k_0 *are constants (the acceleration due to gravity and the per unit mass coefficient of friction respectively). The function*

$$x(t) = e^{-k_o t} + \frac{g}{k_0}$$

is a solution for all t. *To see this note*

$$x'(t) = -k_0 e^{-k_0 t}$$

is equal to

$$f(t, x(t)) = g - k_0 \left(e^{-k_o t} + g/k_0\right)$$

for all t.

For any constant c *the function*

$$x(t) = ce^{-k_o t} + \frac{g}{k_0}$$

is also solution for all t *since*

$$x'(t) = -k_0 ce^{-k_0 t}$$

is equal to

$$f(t, x(t)) = g - k_0 \left(ce^{-k_o t} + \frac{g}{k_0}\right)$$

for all t. *In Chapter 2 it is shown that this formula is in fact the general solution.*

The solution satisfying the initial condition $v(0) = 0$ *(which describes an object that is initially dropped) is found from the general solution by solving*

$$x(0) = c + \frac{g}{k_0} = 0$$

for

$$c = -\frac{g}{k_0}.$$

This yields the solution

$$x(t) = -\frac{g}{k_0} e^{-k_o t} + \frac{g}{k_0}.$$

In applications solutions are not always defined for all t. Here is an example.

EXAMPLE 1.2. *An equation describing the growth of the world's human population* $x(t)$ *in billions as a function of time* t *(in years) is*

$$x' = kx^{p+1}$$

where $k > 0$ *and* $p > 0$ *are positive constants estimated from data (see Chapter 3, Sec. 6.) The function*

$$x(t) = \frac{1}{(1 - pkt)^{\frac{1}{p}}}$$

is defined on the interval $t < 1/pk$. (The denominator vanishes at $t = 1/pk$.) This function is a solution for $t < 1/pk$ since

$$x'(t) = k\frac{1}{(1-pkt)^{\frac{p+1}{p}}}$$

and

$$f(t, x(t)) = k\left(\frac{1}{(1-pkt)^{\frac{1}{p}}}\right)^{p+1} = k\frac{1}{(1-pkt)^{\frac{p+1}{p}}}$$

are identically equal for all $t < 1/pk$.

Similar calculations show the function

$$x(t) = \frac{x_0}{(1-pkx_0^p t)^{\frac{1}{p}}}$$

is a solution on the interval $t < 1/pkx_0^p$ for any constant $x_0 > 0$. This solution satisfies the initial condition $x(0) = x_0$.

A formula for the general solution of an equation $x' = f(t, x)$ cannot always be found. The right hand side of the equation $f(t, x)$ involves the unknown solution x and therefore is not a known function of t that we can integrate. Nonetheless, the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{1.3}$$

has one and only one solution under appropriate conditions placed on $f(t, x)$ as a function of t and x. The derivative of $f(t, x)$ with respect to x is denoted by $\partial f(t, x)/\partial x$ (and called the "partial derivative" of f with respect to x).

THEOREM 1.1. ***(Fundamental Existence and Uniqueness Theorem)*** *Suppose $f(t, x)$ and its derivative $\partial f(t, x)/\partial x$ with respect to x are continuous for x near x_0 and t near t_0*[2]*. Then the initial value problem (1.3) has a solution on an interval containing t_0. Moreover, there is no other solution of the initial value problem on this interval.*

For example, consider the initial value problem

$$x' = tx, \quad x(0) = \frac{1}{2}.$$

The function

$$f(t, x) = tx$$

and its derivative

$$\frac{\partial f(t, x)}{\partial x} = t,$$

are continuous for all x and t (and therefore, certainly for x near $x_0 = 1/2$ and t near $t_0 = 0$). Therefore, by Theorem 1.1 this initial value problem has a unique solution on an interval containing $t_0 = 0$. (From the formula $x(t) = e^{t^2/2}/2$ for the

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[2] By "continuous for x near x_0" we mean continuous on an interval $a < x_0 < b$ containing x_0. Similarly, by "continuous for t near t_0" we mean continuous on an interval $c < t_0 < d$ containing t_0.

solution it is seen that the solution is defined for all t, a fact not obtainable from Theorem 1.1.)

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EXAMPLE 1.3. *An initial value problem describing the growth of a population in a periodically fluctuating environment is*

$$x' = rx\left(1 - \frac{x}{K + a\sin t}\right), \quad x(0) = x_0$$

where x_0 *is the initial population size and* r, K *and* $a < K$ *are positive constants. Since the denominator never vanishes the function*

$$f(t,x) = r\left(x - \frac{x^2}{K + a\sin t}\right)$$

and its derivative with respect to x

$$\frac{\partial f(t,x)}{\partial x} = r\left(1 - 2\frac{x}{K + a\sin t}\right)$$

are continuous for all x *and* t. *Therefore, the initial value problem has a unique solution on an interval containing* $t_0 = 0$. *No algebraic formula is available for the general solution of this equation, nor for the solution of initial value problems.*

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If one or both of the conditions on $f(t,x)$ in the existence and uniqueness Theorem 1.1 fail to hold, then one can draw no conclusions from this theorem. In particular, one *cannot* conclude in this case that there is not a solution. For example, for the initial value problem

$$x' = x^{1/3}, \quad x(0) = 0 \tag{1.4}$$

the function

$$f(t,x) = x^{1/3}$$

fails to satisfy the conditions in Theorem 1.1 because the derivative

$$\frac{\partial f(t,x)}{\partial x} = \frac{1}{3}x^{-2/3},$$

is not continuous at $x_0 = 0$ (it is not even defined there). Yet this initial value problem does have a solution: $x(t) = 0$. For an example of an initial value problem that has no solution see Exercise 1.25.

The initial value problem (1.4) also provides an example of non-uniqueness since $x(t) = 0$ and

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}$$

are two different solutions. This does not contradict Theorem 1.1 because the theorem does not apply to this initial value problem.

The Fundamental Existence and Uniqueness Theorem 1.1 provides criteria under which an initial value problem has a solution on an interval containing the initial point $t = t_0$. The *maximal interval* of the solution is the largest interval containing t_0 on which it solves the differential equation. Theorem 1.1 gives no information about the maximal interval of a solution. In fact, without a solution formula it is usually difficult to determine the maximal interval. The function $f(t,x)$ may satisfy the criteria of Theorem 1.1 for all values of t and x and yet solutions may not be defined for all t.

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EXAMPLE 1.4. *Consider the initial value problem*

$$x' = 2tx^2, \quad x(0) = 1.$$

The function

$$f(t,x) = 2tx^2$$

and its derivative

$$\frac{\partial f(t,x)}{\partial x} = 4tx$$

are continuous for all x *and* t. *Theorem 1.1 implies there exists a unique solution on an interval containing* $t_0 = 0$. *The solution formula*

$$x(t) = \frac{1}{1-t^2}$$

shows the maximal interval is $-1 < t < 1$. *See Fig. 1.1.*

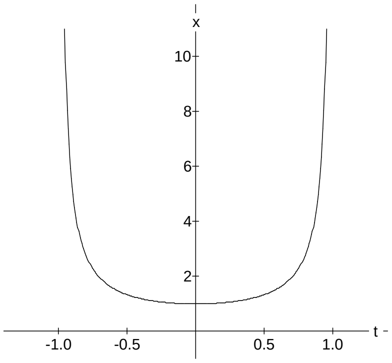

FIGURE 1.1 The solution of the initial value problem $x' = 2tx^2$, $x(0) = 1$ has vertical asymptotes at $t = \pm 1$.

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The importance of the interval of existence of a solution can sometimes be overlooked. Here is an example.

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EXAMPLE 1.5. *A popular computer program gives the formula* $x(t) = \sin t$ *for the solution of the initial value problem*

$$x' = \sqrt{1-x^2}, \quad x(0) = 0$$

without indicating the solution interval. Since $\sin t$ *is defined for all* t, *the implication is that* $\sin t$ *is a solution for all* t. *This is false, however, since*

$$x'(t) = \cos t$$

and

$$f(t, x(t)) = \sqrt{1 - \sin^2 t}$$

are equal only on intervals where $\cos t$ *is positive. Thus, the formula* $x(t) = \sin t$ *defines a solution on the interval* $-\pi/2 < t < \pi/2$, *but on no larger interval containing* $t_0 = 0$. *(However, this interval is not the maximal interval of the solution of the initial value problem! See Exercise 1.26.)*

EXERCISES

Find the general solution of the following differential equations.

Exercise 1.1. $x' = 1 + t^2$

Exercise 1.2. $x' = \cos \pi t$

Exercise 1.3. $x' = e^{2t}$

Exercise 1.4. $x' = te^{-t}$

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Find the unique solution of the following initial value problems.

Exercise 1.5. $x' = t^2$, $x(1) = 2$

Exercise 1.6. $x' = e^{-3t}$, $x(0) = 1$

Exercise 1.7. $x' = te^{-t}$, $x(0) = 1$

Exercise 1.8. $x' = \sin 3t$, $x\left(\frac{\pi}{6}\right) = 0$

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For which initial value problems can the Fundamental Existence and Uniqueness Theorem 1.1 be applied? Explain your answer. In each case, what do you conclude from this theorem?

Exercise 1.9. $x' = t^2 + x^2$, $x(0) = 0$

Exercise 1.10. $x' = \frac{t^2}{x^2}$, $x(0) = 0$

Exercise 1.11. $x' = \tan x$, $x\left(\frac{\pi}{2}\right) = 0$

Exercise 1.12. $x' = \tan x$, $x(0) = 0$

Exercise 1.13. $x' = \tan x$, $x(0) = \frac{\pi}{2}$

Exercise 1.14. $x' = \ln(tx)$, $x(1) = 2$

Exercise 1.15. $x' = \frac{1}{\sin x}$, $x(0) = \frac{\pi}{2}$

Exercise 1.16. $x' = \frac{1}{t-x}$, $x(-1) = 2$

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For what values of the constant a can the Fundamental Existence and Uniqueness Theorem 1.1 be applied to the initial value problems below? Explain your answer. What do you conclude from this theorem for such values of a? What do you conclude from this theorem for other values of a?

EXERCISE 1.17. $x' = \ln(a - x)$, $x(0) = 0$

EXERCISE 1.18. $x' = \tan ax$, $x(0) = \frac{\pi}{2}$

EXERCISE 1.19. $x' = \sqrt{a^2 - x^2}$, $x(1) = 2$

EXERCISE 1.20. $x' = \frac{1}{a-x}$, $x(1) = 2$

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For which t_0 and x_0 does the Fundamental Existence and Uniqueness Theorem 1.1 apply to the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$, with the functions $f(t,x)$ below? Explain your answer. What do you conclude from this theorem for such initial points? What do you conclude from this theorem for other initial points?

EXERCISE 1.21. $f = \ln\left(t^2 + x^2\right)$

EXERCISE 1.22. $f = \frac{t^2}{x}$

EXERCISE 1.23. $f = \tan bx$, $b =$ *constant*

EXERCISE 1.24. $f = \sqrt{t^2 + x^2 - b^2}$, $0 < b =$ *constant*

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EXERCISE 1.25. *Consider the initial value problem* $x' = f(t,x)$, $x(0) = 0$ *where*

$$f(t,x) = \begin{cases} 1, & \text{for } t \geq 0 \text{ and all } x \\ -1, & \text{for } t < 0 \text{ and all } x. \end{cases}$$

(a) Show the existence and uniqueness Theorem 1.1 does not apply. What do you conclude?

(b) Show this initial value problem does not have a solution on any interval containing $t_0 = 0$.

EXERCISE 1.26. *Consider the equation* $x' = \sqrt{1 - x^2}$.

(a) Show the constant functions $x(t) = 1$ *and* $x(t) = -1$ *are solutions for all* t.

(b) Show the function

$$x(t) = \begin{cases} 1 & \text{for } t \geq \frac{\pi}{2} \\ \sin t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ -1 & \text{for } t \leq -\frac{\pi}{2} \end{cases}$$

is a solution for all t. *Thus, the maximal interval for the solution of the initial value problem* $x(0) = 0$ *is the whole real line.*

(c) The solution $x(t) = 1$ *and the solution in (b) both satisfy the same initial value problem* $x(\pi/2) = 1$ *for all* t. *Why does this not contradict Theorem 1.1?*

2. Approximation of Solutions

Formulas for solutions of differential equations are not in general available. For this reason we need other methods for studying equations and their solutions. For some applications it is sufficient to obtain approximations to solutions. For example, roughly sketched graphs of solutions are sometimes adequate. In other applications, more accurate graphs or even numerical approximations are necessary. One can also obtain algebraic formulas for approximations to solutions. In this section we study some graphical and numerical approximation methods. Analytic approximation methods are studied in Chapter 3. We begin with a procedure for making sketches of solution graphs.

2.1. Slope Fields. From algebra and calculus we learn that graphs are a useful way to study functions. The derivative of a function is the slope of its graph. A differential equation therefore tells us something about the slopes of the graphs of its solutions.

Specifically, if the graph of a solution $x = x(t)$ of

$$x' = f(t,x) \tag{2.1}$$

passes through a point (t,x), then the slope $x'(t)$ of its graph at this point equals $f(t,x(t))$. In other words, each point (t,x) in the domain of f is associated with a slope equal to the number $f(t,x)$.

For example, the graph of a solution of $x' = t^2+x^2$ that passes through the point $(t,x) = (1,1)$ necessarily has slope $1^2+1^2 = 2$ at this point. Similarly, the solution whose graph passes through the point $(-2,1/3)$ must have slope $(-2)^2+(1/3)^2 = 37/9$ at this point.

The association of a slope $f(t,x)$ with each point (t,x) defines the *slope field* of the differential equation (2.1). Solutions of differential equation must "fit" its slope field. This means at each point on a solution's graph the slope (of the tangent) must equal the slope associated with that point.

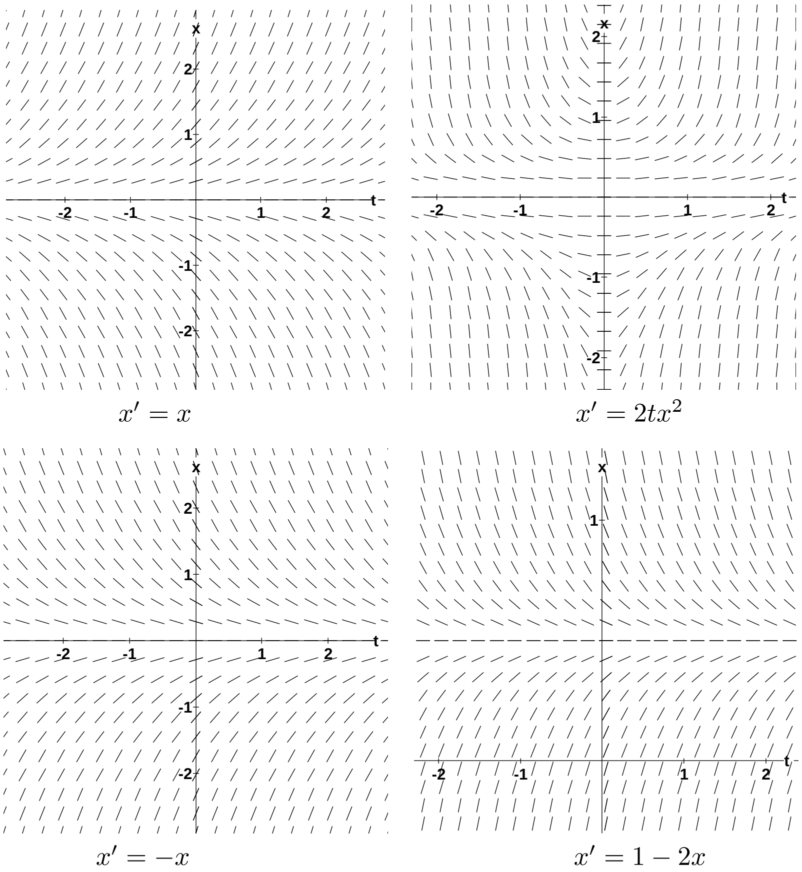
One way to obtain a picture of a slope field is to draw, through each of several points in the (t,x)-plane, a short straight line segment that has the slope associated with that point. By drawing such line segments through a sufficient number of points in the plane, we can get a good approximation to the overall slope field and hence the graphs of solutions.

Rather than randomly choosing points in the plane, it is better to proceed in a systematic manner. We discuss two ways to do this: the "grid" and the "isocline" methods. The grid method is particularly well suited for computer use. The isocline method is sometimes a convenient way to obtain a sketch of the slope field by hand.

THE GRID METHOD

One way to approximate a slope field is to draw a short line segment with the appropriate slope at points lying on a rectangular grid in the (t,x)-plane. This *grid method* can be done by hand; however, most computer programs that "solve" differential equations will also draw slope fields using this "grid" method and display the results graphically.

When sketching a slope field by the grid method, one must chose a grid fine enough so that the essential features of the slope field are apparent, but coarse enough so as not to be visually cluttered. It usually takes a several attempts to find a suitable gird size. Sample slope fields for several differential equations, drawn using the grid method, appear in Fig. 2.1.



$x' = x$ $\qquad$ $x' = 2tx^2$

$x' = -x$ $\qquad$ $x' = 1 - 2x$

FIGURE 2.1. Slope fields are shown for four different differential equations.

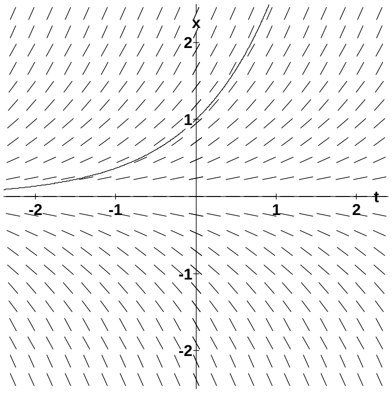


FIGURE 2.2 The slope field for $x' = x$ and the solution satisfying the initial condition $x(0) = 1$.

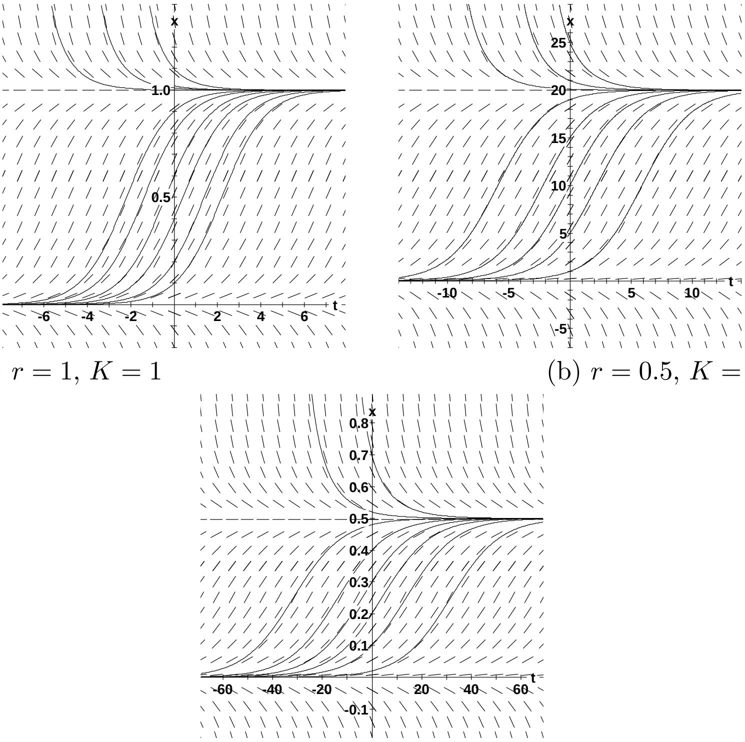
One can sketch the solution graph of an initial value problem $x(t_0) = x_0$ by drawing a curve that both fits the slope field and passes through the point (t_0, x_0). Such a sketch can often suggest important properties of solutions. For example, the slope field and solution sketched in Fig. 2.2 suggest that the solution is monotonically increasing without bound as $t \to +\infty$ and that the x-axis is a horizontal asymptote as $t \to -\infty$.

The next example shows how a slope field can yield important general properties of solutions.

EXAMPLE 2.1. *Fig. 2.3 shows the slope fields of the logistic equation*

$$x' = rx\left(1 - \frac{x}{K}\right)$$

for several choices of the parameters r and K. These slope fields, together with the sample solution graphs, suggest that solutions with positive initial conditions $x(0) = x_0 < K$ tend monotonically to a horizontal asymptote at $x = K$ as $t \to +\infty$. This important fact about the logistic equation will be proved in Chapter 3. Note that $x(t) = K$ is a solution.



(a) $r = 1$, $K = 1$

(b) $r = 0.5$, $K = 20$

(c) $r = 0.1$, $K = 0.5$

FIGURE 2.3. Selected slope fields and solutions for the logistic equation $x' = r\,(1 - x/K)\,x$.

THE ISOCLINE METHOD

In Fig. 2.3 it is interesting to note that the points lying on a horizontal straight line appear to be associated with the same slope. The reason for this is that $f(t, x) = rx(1 - x/K)$, and hence the slope at a point (t, x), does not depend on t.This observation in fact applies to any equation whose right hand side f does not depend on the independent variable t, i.e. to any so-called "autonomous" equation (Chapter 3).

A curve all of whose points are associated with the same slopes in the slope field of a differential equation is called an *isocline.* ("iso" means "same" and "cline" means "slope" .) The isoclines of an autonomous equation $x' = f(x)$ are horizontal straight lines. Points on a horizontal line $x = a$ are associated with slope $f(a)$. This fact can be a useful aid in sketching the slope field of an autonomous equation. Fig. 2.4 shows a sketch of the slope field for the equation $x' = x(1 - x)$ obtained using this isocline method.

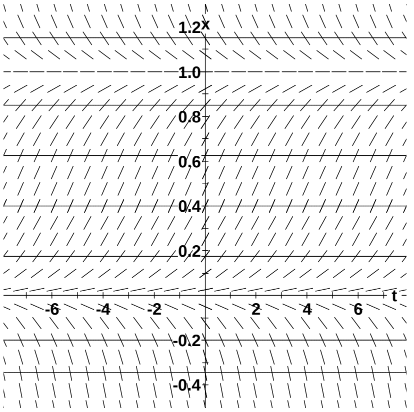


FIGURE 2.4. Some isoclines for $x' = x\,(1 - x)$ are shown.

The concept of an isocline is not restricted to autonomous equations. For any equation $x' = f(t, x)$ we can find isoclines by determining those points in the plane that are associated with a common slope m. These points satisfy the *isocline equation*

$$f(t, x) = m.$$

The graph of this equation is, in general, a curve in the plane called the *isocline* associated with slope m.

For nonautonomous equations isoclines are not necessarily horizontal lines. If they can be conveniently graphed, isoclines can be used to sketch slope fields for nonautonomous equations in the same way they were used for autonomous equations. On an isocline we draw several short line segments each having the slope associated with that isocline. Doing this for a collection of isoclines we obtain a sketch of the slope field. The following example illustrates the method.

EXAMPLE 2.2. *What are the isoclines associated with the equation*

$$x' = t^2 + x^2 \ ?$$

Suppose we find the isocline associated with slope $m = 1$. *The equation for this isocline is* $t^2 + x^2 = 1$ *which we recognize as the equation the circle with radius* 1 *and center at the origin* $(0, 0)$. *Drawing this circle and placing on it several short line segments with slope* 1, *we obtain part of the slope field. This procedure can be repeated using other slopes* m. *Points associated with slope* $m = 2$ *lie on the circle of radius* $\sqrt{2}$ *while points associated with slope* $m = 0.25$ *lie on the circle of radius* $\sqrt{0.25}$ *and so on. The typical isocline equation* $t^2 + x^2 = m$ *yields the circle of radius* $\sqrt{m}$, *provided* $m > 0$. *A "degenerate" isocline is obtained for slope* $m = 0$, *namely the single point* $(0, 0)$. *There are no isoclines associated with negative slopes* $m < 0$. *See Fig. 2.5(a).*

Isoclines are not necessarily easy to identify or graph. Their usefulness for slope field sketching depends on the right hand side $f(t, x)$ of the differential equation. If we can easily identify and graph isoclines, then this method for drawing slope fields is convenient. Otherwise it is not.

Caution: A common mistake is to confuse isoclines with the solution graphs. Isoclines are *not* graphs of solutions. For example, compare the solution graph in Fig. 2.5(b) to the circular isoclines in Fig. 2.5(a).

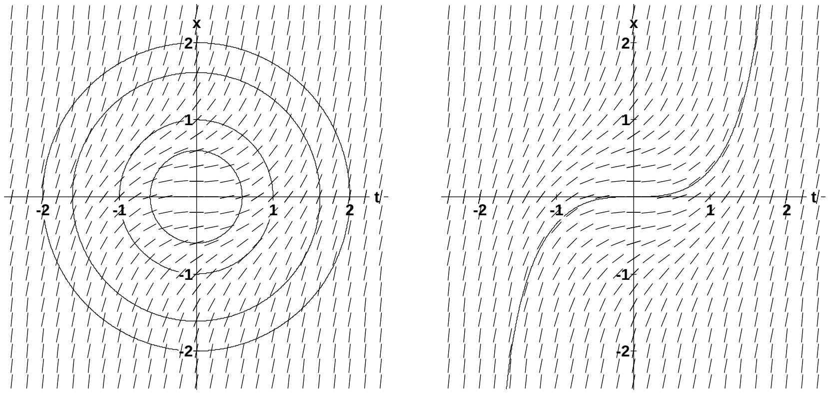


FIGURE 2.6. (a) Selected circular isoclines of $x' = t^2 + x^2$. (b) The solution satisfying the initial condition $x(0) = 0$.

EXERCISES

Use a computer to obtain sketches of the slope fields for the differential equations in the exercises below. Using the slope field, sketch (by hand) the graph of the solutions satisfying each of the given initial conditions.

EXERCISE 2.1. $x' = 1 - x$
$x(0) = 3, \quad x(0) = 0, \quad x(-1) = 2$

EXERCISE 2.2. $x' = 2 - 3x$
$x(0) = 1, \quad x(0) = \frac{2}{3}, \quad x(0) = -1$

EXERCISE 2.3. $x' = 1 - x^2$
$x(0) = -1, \quad x(-2) = 1, \quad x(0) = 0, \quad x(1) = 1.2$

EXERCISE 2.4. $x' = x\left(1 - \frac{x}{2+\cos t}\right)$
$x(0) = 0, \quad x(0) = 1, \quad x(0) = -0.1, \quad x(-2) = 2$

EXERCISE 2.5. $x' = x\cos t$
$x(0) = 0, \quad x(1) = 4, \quad x(0) = 1, \quad x(0) = -1$

EXERCISE 2.6. $x' = -\frac{1}{2}x + \sin t$
$x(0) = 0, \quad x(0) = 1, \quad x(-2) = -1, \quad x(0) = -1$

EXERCISE 2.7. $x' = x\sin x$
$x(0) = 0, \quad x(0) = \frac{\pi}{2}, \quad x(0) = -3, \quad x(0) = 4$

EXERCISE 2.8. $x' = \left(1 + t^2 + x^2\right)^{-1/2}$
$x(0) = 0, \quad x(-1) = -1.5$

EXERCISE 2.9. $x' = (1-x)\,x\sin^2 t$
$x(0) = -0.25, \quad x(0) = 2, \quad x(-2) = 0.5$

EXERCISE 2.10. $x' = (1-x^2)\,(\sin t - x)$
$x(0) = 0, \quad x(0) = -0.5, \quad x(1) = 1.5, \quad x(0) = 1$

EXERCISE 2.11. $x' = x(1-x)(x+1)$
$x(0) = 0.5, \quad x(0) = -0.5, \quad x(0) = 1.5, \quad x(0) = -1.5$

EXERCISE 2.12. *Consider the differential equation in Example 1.3 :*

$$x' = rx\left(1 - \frac{x}{K + a\sin t}\right).$$

(a) Use a computer to sketch the slope fields of the equation in the window $0 \le t \le 20$*,* $0 \le x \le 10$ *for the cases below.*

(i) $r = 1, \quad K = 2, \quad a = 1$

(ii) $r = 1, \quad K = 5, \quad a = 1$

(iii) $r = 0.5, \quad K = 5, \quad a = 2$

(iv) $r = 0.5, \quad K = 5, \quad a = 4$

(b) For each case in (a), use the slope field to sketch (by hand) the graphs of the solutions satisfying the initial condition $x(0) = 1$*.*

(c) What do all the solutions graphed in (b) seem to have in common?

Use a computer to obtain a sketch of the slope field for the equations below. Do this for a selection of values for the constant a. How are the slope fields for $a > 1$ different from those for $a < 1$?

EXERCISE 2.13. $x' = -a + 2x - x^2$

EXERCISE 2.14. $x' = x(x-a)(1-x)$

Describe (geometrically) and sketch the isoclines for the differential equations below and use them to obtain a sketch of the slope fields.

EXERCISE 2.15. $x' = 1 - x$

EXERCISE 2.16. $x' = 4 - 2x$

EXERCISE 2.17. $x' = \left(1 + t^2 + x^2\right)^{-1/2}$

EXERCISE 2.18. $x' = -x + \sin t$

————————————————————————————————

Find first order differential equations whose isoclines are as described below, if possible. Here m denotes the slope in the field slope. If there is no such equation, explain why.

EXERCISE 2.19. *The family of lines* $x = t + m$ *where* m *allowed to be any constant.*

EXERCISE 2.20. *The family of parabolas* $x = t^2 + m$ *where* m *is allowed to be any constant.*

EXERCISE 2.21. *The family of lines* $x = t + \frac{1}{m}$ *where* m *is allowed to be any nonzero constant.*

EXERCISE 2.22. *The family of parabolas* $x = t^2 + \frac{1}{m}$ *where* m *is allowed to be any nonzero constant.*

EXERCISE 2.23. *The family of ellipses* $2x^2 + 3t^2 = m^{1/3}$ *where* m *is allowed to be any positive constant.*

EXERCISE 2.24. *The family of circles* $x^2 + t^2 = 1 - 2m^2$ *where* m *is allowed to be any positive constant satisfying* $0 < c < 1/\sqrt{2}$.

2.2. Numerical and Graphical Approximations. Slope fields provide approximate graphs of solutions of differential equations. However, it is often desirable to have a more accurate approximation to a solution and its graph than can be obtained from a slope field. Another way to obtain an approximate graph of a solution on an interval $t_0 \leq t \leq T$ is to calculate numerical approximations x_i to the solution $x(t_i)$ at $t = t_i$ where

$$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T.$$

and, in the (t, x)-plane, connect the points (t_0, x_0), (t_1, x_1),..., (t_n, x_n) by straight line segments. See Fig. 2.6.

We want to obtain the approximations $x_i \approx x(t_i)$ in such a way that if the number of points t_i increases (and the distances between them tend to zero) then the approximations x_i become more accurate and the approximate ("broken line") graphs approach the (smooth) graph of the solution $x = x(t)$.

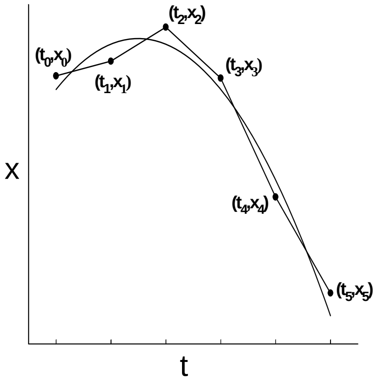


FIGURE 2.6. A "broken line" approximate graph obtained from approximations x_i to the solution at t_i.

In this section we study a basic method for approximating the solution of the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{2.2}$$

at specified values of $t > t_0$. The method, called the *Euler Algorithm*, is a fundamental method that serves as an introduction to the numerical approximation of solutions of differential equations. It is, however, rarely used for other than pedagogical reasons because it "converges" too slowly. Sec. 2.3 gives some methods that converge more quickly (and hence are more commonly used). Nonetheless Euler's Algorithm, by providing a basis for understanding how solutions are numerically approximated, is a good starting point for the study of more efficient (and hence complicated) algorithms.

Consider the problem of approximating the solution $x = x(t)$ of (2.2) at $t = t_1 > t_0$. Since $x(t)$ is a solution, we can integrate both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_0$ to $t = t_1$ to obtain

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(t, x(t))dt$$

or, using the initial condition,

$$x(t_1) = x_0 + \int_{t_0}^{t_1} f(t, x(t))dt. \tag{2.3}$$

The right hand side of this equation does not give a formula for $x(t_1)$ because it involves the unknown solution $x(t)$. However, we can use (2.3) to approximate $x(t_1)$ by making an approximation to the integral on the right hand side. For example, we can use integration approximation methods studied in calculus, such as the rectangle rule, the trapezoid rule, or Simpson's rule.

The Euler Algorithm is obtained by using the (left hand) rectangle rule to approximate the integral :

$$\int_{t_0}^{t_1} f(t, x(t))dt \approx (t_1 - t_0) f(t_0, x(t_0)).$$

Defining the first step size by $s_0 = t_1 - t_0$ and recalling the initial condition $x(t_0) = x_0$ we have

$$\int_{t_0}^{t_1} f(t, x(t))dt \approx s_0 f(t_0, x_0)$$

and consequently from (2.3) we have the approximation

$$x(t_1) \approx x_0 + s_0 f(t_0, x_0).$$

Denote this approximation by x_1; that is, we define x_1 by

$$x_1 = x_0 + s_0 f(t_0, x_0).$$

To obtain an approximation x_2 to the solution value $x(t_2)$ at the next point t_2 we proceed in a similar manner. Integrate both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_1$ to $t = t_2$. Using the Fundamental Theorem of Calculus, the (left hand) rectangle rule to approximate the integral and the approximation $x_1 \approx x(t_1)$, we obtain

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} f(t, x(t))dt \approx x_1 + (t_2 - t_1) f(t_1, x_1)$$

We denote this approximation to the solution at $t = t_2$ by

$$x_2 = x_1 + s_1 f(t_1, x_1), \quad s_1 = t_2 - t_1.$$

In calculating the approximation x_2 we introduced *two* sources of error. First, there is the error made in using the rectangle rule to approximate the integral (called the "truncation error") and, secondly, there is the error in using the approximation x_1 to $x(t_1)$. Together these errors account for the "accumulation error" at the point $t = t_2$.

If this procedure is repeated we obtain the following formulas

$$\begin{aligned} x_0 &= x_0 \\ x_{i+1} &= x_i + s_i f(t_i, x_i), \qquad s_i = t_{i+1} - t_i, \quad i = 0, 1, 2, \cdots \end{aligned}$$

of the Euler Algorithm. The number x_i is an approximation to the solution $x = x(t)$ of the initial value problem (2.2) at the point $t = t_i$. Usually equally spaced points are chosen, in which case $s_i = s$ for all i and the algorithm reduces to

$$\begin{aligned} x_0 &= x_0 \\ x_{i+1} &= x_i + sf(t_i, x_i) \quad \text{for} \quad i = 0, 1, 2, \cdots, n. \end{aligned} \tag{2.4}$$

The common distance s is called the *step size* of the algorithm.

The formulas (2.4) are recursive. That is to say, one utilizes the same formula sequentially to calculate the approximations at each of the points $t_1, t_2, \dots t_n$, using at each step the approximation made at the previous step. This makes the method ideally suited for programming on a computer or calculator.

The accuracy of the integral approximation obtained by the rectangle rule increases if the step size s decreases. For this reason we expect the accuracy of the approximations obtained from the Euler Algorithm (2.4) to increase if the step size s decreases. There is a cost for this increased accuracy, however, because decreasing the step size s will increase the number n of steps necessary to get from the initial condition t_0 to the end point T. This means more repetitions of the algorithm (2.4) are required, and consequently more arithmetic work is necessary to reach the end point $t_n = T$. (This also means more round off errors!)

– –

Example 2.3. *In this example we use the Euler Algorithm (2.4) to approximate the solution* $x = x(t)$ *of the initial value problem*

$$x' = x, \quad x(0) = 1$$

at $T = 1$ *using step size* $s = 0.2$*. The Euler algorithm (2.4) for this problem is*

$$x_{i+1} = x_i + sx_i \quad \textit{for} \quad i = 0, 1, 2, \cdots$$

with $x_0 = 1$*. Using step size* $s = 0.2$ *we need to calculate approximations at the five points* $t = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$*. The calculations are*

$$\begin{aligned} x_1 &= x_0 + sx_0 = 1 + 0.2 \times 1 = 1.2 \\ x_2 &= x_1 + sx_1 = 1.2 + 0.2 \times 1.2 = 1.44 \\ x_3 &= x_2 + sx_2 = 1.44 + 0.2 \times 1.44 = 1.728 \\ x_4 &= x_3 + sx_3 = 1.728 + 0.2 \times 1.728 = 2.0736 \\ x_5 &= x_4 + sx_4 = 2.0736 + 0.2 \times 2.0736 = 2.48832. \end{aligned}$$

The Euler Algorithm with step size $s = 0.2$ *yields the approximation* $x(1) \approx x_5 = 2.48832$.

– –

How good is the approximation x_5 in the previous example? More generally, how accurate are the approximations (2.4) of the Euler Algorithm? Can we estimate the size of the error and if not how can we have any confidence in the numerical approximations obtained from the formulas (2.4)?

An accurate estimate of the error resulting from approximation methods such as the Euler Algorithm is usually not possible. However, we expect the numerical approximations will get more accurate as the step size s decreases and that they will tend to the exact solution in the limit as $s \to 0$. This turns out to be true for the Euler Algorithm, on the solution's interval of existence, under the assumptions of the Fundamental Existence and Uniqueness Theorem 1.1.

One useful way to study the accuracy of the Euler Algorithm (and of other algorithms as well) is to consider the *rate* at which the approximations converge to the exact solution. The Euler Algorithm is said to be "first order" or of "order 1". What this means is that the magnitude of the error at $t = T$ is no larger than constant multiple of the first power of s. That is to say, there exists a constant $c > 0$ such that $\mid x(T) - x_n \mid \leq cs$. This inequality guarantees the Euler approximations converge to the value of the solution at least as fast as s decreases to 0. Thus, roughly speaking, if the step size s is halved, then in general we expect the error to be (at least) halved. If the step size is decreased by a factor of 1/10, then in general we expect the error to decrease by a factor of 1/10 and so on. (For an example, see Table 2.2 below.) We summarize this by saying that the Euler Algorithm is "$O(s)$" (pronounced "Oh of s").

We can gain confidence in the accuracy of numerical approximations by observing their changes as the step size s decreases. This is commonly done by decreasing s by a fixed fraction. For example, if s is decreased by one half several times, we expect the error to be cut in half each time. Since the approximations at a fixed t approach the solution value $x(t)$, the leading digits in the resulting sequence of approximations should eventually "stabilize" (i.e., remain unchanged as s decreases further). As a practical matter we accept these digits as correct. However, none of these digits may be accurate, since we cannot be sure that they will remain unchanged if the step size s decreases further.

Example 2.4. *In this example we repeat Example 2.3 by halving the step size s six consecutive times and observe the resulting change in the approximation to* $x(1)$. *The number of calculations necessary to perform the approximation increases as s decreases. For example, the algorithm (2.4) must be used* 320 *times for the step size* $s = 0.003125$.

We use a computer to perform the calculations and the results appear in Table 2.1. We expect the approximation $x(1) \approx 2.714047$ *obtained from the smallest step size* $s = 0.003125$ *to be the most accurate, but how many of these digits are correct? We know the sequence of approximations converges to the exact value of the solution at* $T = 1$. *Since only two digits appear to have stabilized in Table 2.1, we accept only the two digit approximation* 2.7 *as accurate.*

Step size s	Approximation to $x(1)$
0.200000	2.488320
0.100000	2.593742
0.050000	2.653298
0.025000	2.685064
0.012500	2.701485
0.006250	2.709836
0.003125	2.714047

Table 2.1. The Euler Algorithm approximations to the solution at $t = 1$ of the initial value problem $x' = x$, $x(0) = 1$ obtained by repeatedly halving the step size.

There is a formula for the solution of the initial value problem in Examples 2.3 and 2.4, namely $x(t) = e^t$. Therefore, the exact value of the solution at $t = 1$ is

$x(1) = e$ (recall $e \approx 2.718282$). Using this formula we can investigate how accurate the approximations in Table 2.1 really are.

Step size s	Approximation to $x(1)$	% Error
0.200000	2.488320	8.4598
0.100000	2.593742	4.5816
0.050000	2.653298	2.3906
0.025000	2.685064	1.2220
0.012500	2.701485	0.6179
0.006250	2.709836	0.3107
0.003125	2.714047	0.1558

TABLE 2.2. The percent errors of the approximations in Table 2.1.

The percent error of each approximation is given in Table 2.2. Notice the percent error decreases by a factor of (approximately) 1/2 at each consecutive step. This is what we expect, since the step size s decreases by a factor of 1/2 at each step and the Euler Algorithm is $O(s)$.

We approximate the graph of solution of the initial value problem $x' = x$, $x(0) = 1$ by connecting the points (t_i, x_i) with straight line segments. This is done in Fig. 2.7 for decreasing step sizes on the interval $0 \leq t \leq 5$. The convergence, as s decreases, of these approximate graphs to the graph of the solution $x = e^t$ is apparent.

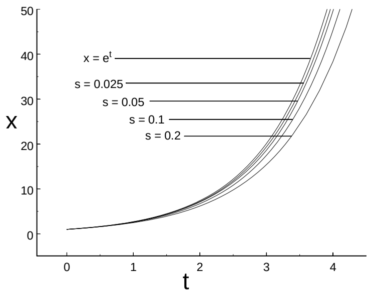


FIGURE 2.7. The broken line graphs calculated from approximations using Euler's Algorithm converge to the solution of the initial value problem $x' = x$, $x(0) = 1$ as the step size s decreases.

One should not accept a graphical approximation to a solution obtained from a single step size s alone (e.g., the default step size in a computer program). Instead, before accepting a graphical approximation, one should decrease the step size until little change occurs in two consecutive graphical approximations.

EXAMPLE 2.5. *The equation* $x' = ae^{-bt}x$ *describes the growth of a tumor where* $x = x(t)$ *is a measure of its size (e.g., weight or number of cells) and* t *is time. Fig. 2.8 shows approximate graphs of the solution of the initial value problem with* $x_0 = 5$ *and parameter values* $a = 20$ *and* $b = 15$. *These graphs result from the Euler Algorithm using a decreasing sequence of step sizes starting with* $s = 0.1$. *Little change occurs in the graphs for the last two steps sizes* $s = 0.003125$ *and* 0.0015625 *and therefore we accept the final graph as an accurate approximation. All* *of the graphs indicate that the tumor size x approaches a maximal size as $t \to +\infty$. However, the inaccurate graphs obtained from the larger steps sizes considerably over estimate the maximal size of the tumor.*

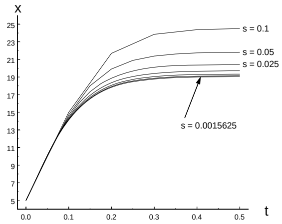


FIGURE 2.8 The Euler Algorithm with a decreasing sequence of steps sizes yields converging approximate graphs for the solution of the initial value problem $x' = 20e^{-15t}x$, $x(0) = 5$.

The convergence rate $O(s)$ of the Euler Algorithm is sometimes too slow for practical purposes. In Table 2.2 only two digits of accuracy for $x(1)$ are obtained with a step size $s = 0.003125$. To obtain more accuracy a smaller step size is needed. However, there are more intermediate steps with each decrease in step size and it takes longer to perform all of the necessary calculations. Furthermore, other sources of error, such as round-off errors at each step, might eventually prevent increased accuracy if the number of steps (and hence calculations) becomes too large.

Table 2.3 shows an example that dramatically illustrates the slow convergence of the Euler Algorithm. In this example no accurate digits are found with a step size as small as $s = 0.000391$.

Step size s	Approximation to $x(1)$
0.100000	5.862897
0.050000	8.905711
0.025000	13.766320
0.012500	21.242856
0.006250	31.967263
0.003125	45.709606
0.001563	60.736659
0.000781	74.330963
0.000391	84.517375

TABLE 2.3. Euler Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/\left(100 - 99t\right)$ for this initial value problem gives the exact value $x(1) = 99$.[3]

Fortunately, practical algorithms with faster rates of convergent are available. In the following section we discuss algorithms of orders two and four. An algorithm

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[3] The interval of existence for the solution is $-\infty < t < 100/99 \approx 1.0101$. It is interesting to note that the Euler Algorithm will calculate "approximations" at t values outside of this interval. For example, with step size $s = 0.1$, eleven repetitions of the algorithm produce the number $x_{11} = 9.30025$. However, this number cannot be taken as an approximation to the solution at $t = 1.1$ because the solution is not defined at this value of $t > 100/99$.

has *order of convergence* p (or more succinctly is of *order* p), written $O(s^p)$, if the accumulative error is bounded in magnitude by a constant multiple of s^p, i.e., if $\mid x(T) - x_n \mid \leq cs^p$.

To see the advantage of a convergence rate of order greater than $p = 1$ consider an algorithm of order $p = 2$, for which the error satisfies $\mid x(T) - x_n \mid \leq cs^2$. We can expect the error to decrease by a factor of $(1/2)^2 = 1/4$ if the step size s is decreased by a factor of $1/2$, or by a factor of $(1/10)^2 = 1/100$ if the step size s is decreased by a factor of $1/10$, and so on. For an algorithm of order 4 the error decreases even faster, e.g., by a factor of $(1/10)^4 = 1/10000$ if the step size is decreased by a factor of $1/10$.

2.3. Another Numerical Algorithm. In deriving the Euler Algorithm we used the Rectangle Rule to approximate the integral

$$\int_{t_i}^{t_{i+1}} f(t, x(t))dt.$$

More accurate approximations to this integral lead to algorithms that converge faster than the Euler Algorithm. For example, we could use the Trapezoid Rule. (Another choice is Simpson's Rule; see Exercise 2.34.) Integrating both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_i$ to $t = t_{i+1}$ we obtain

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(t, x(t))dt.$$

From the Trapezoid Rule approximation

$$\int_{t_i}^{t_{i+1}} f(t, x(t))dt \approx \frac{s_i}{2}\left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x(t_i))\right]$$

we get

$$x(t_{i+1}) \approx x(t_i) + \frac{s_i}{2}\left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x(t_i))\right].$$

Assuming that we already have an approximation $x(t_i) \approx x_i$ to the solution at the point $t = t_i$, we can write

$$x(t_{i+1}) \approx x_i + \frac{s_i}{2}\left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x_i)\right].$$

Unfortunately we cannot use the right hand side to calculate an approximation x_{i+1} to $x(t_{i+1})$ because it involves $x(t_{i+1})$. This is an example of what is called an *implicit* algorithm because the equation

$$x_{i+1} = x_i + \frac{s_i}{2}\left[f(t_{i+1}, x_{i+1}) + f(t_i, x_i)\right]$$

is not explicitly solved for the approximation x_{i+1}. (The Euler Algorithm is an example of an *explicit* algorithm.) To find the approximation x_{i+1}, we have to solve this equation. To do this at each step results in a highly complicated algorithm. One way to deal with this difficulty is to perform another approximation. For example, we can use the Euler approximation for the x_{i+1} on the right hand side. Thus, at each step we use the formulas

$$\begin{aligned} x_{i+1}^* &= x_i + s_i f(t_i, x_i) \\ x_{i+1} &= x_i + \frac{s_i}{2}\left[f(t_{i+1}, x_{i+1}^*) + f(t_i, x_i)\right], \quad i = 0, 1, 2, \cdots \end{aligned}$$

to calculate the approximation x_{i+1}. This algorithm is called *Heun's Algorithm* (sometimes the *Improved Euler Algorithm* or the *Modified Euler Algorithm*). It is an example of a "predictor-corrector" algorithm. At each step the Euler approximation x^*_{i+1} is the prediction and x_{i+1} is the correction.

If equal step sizes $s_i = s$ are used Heun's Algorithm is

$$\begin{aligned} x^*_{i+1} &= x_i + sf(t_i, x_i) \\ x_{i+1} &= x_i + \frac{s}{2}\left[f(t_{i+1}, x^*_{i+1}) + f(t_i, x_i)\right], \quad i = 0, 1, 2, \cdots . \end{aligned} \tag{2.5}$$

The initial condition x_0 starts the algorithm. It turns out that Heun's Algorithm of order $O(s^2)$.

Compare the results in Table 2.4 with those in Table 2.2. Note that the error in Table 2.4 decreases approximately by a factor of 1/4 as the steps size is decreased by a factor of 1/2. Heun's Algorithm is a popular procedure; for example, it is often used with programmable hand calculators.

Step size s	Approximation to $x(1)$	% Error
0.200000	2.702708	0.5729
0.100000	2.714081	0.1545
0.050000	2.717191	0.0401
0.025000	2.718004	0.0102
0.012500	2.718212	0.0026
0.006250	2.718264	0.0007

TABLE 2.4. The Heun Algorithm approximations to the solution of the initial value problem $x' = x$, $x(0) = 1$, at $t = 1$ obtained by repeatedly halving the step size.

We saw in Table 2.3 an example of an initial value problem for which the Euler Algorithm converges too slowly to be practical. Table 2.5 shows the results of applying Heun's Method to the same initial value problem. The estimates obtained from the two numerical algorithms differ considerably. At each step size Heun's Algorithm provides a more accurate approximation to $x(1) = 99$ than does the Euler Algorithm.

Step size s	Approximation to $x(1)$
0.100000	19.346653
0.050000	33.073325
0.025000	52.217973
0.012500	72.662362
0.006250	87.787581
0.003125	95.273334
0.001563	97.719807
0.000781	98.719804
0.000391	98.928245

TABLE 2.5. Heun's Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.

Even higher order algorithms are available, although they involve more complicated formulas at each step. A widely used class of algorithms are called Runge-Kutta algorithms. These algorithms are available for any order of convergence. A popular algorithm is the fourth order Runge-Kutta algorithm. You can see the complicated formulas for this algorithm in Exercise 2.25. Table 2.6 shows the results applying this algorithm to the same initial value problem in Table 2.3 and 2.5. This faster converging algorithm provides an accurate approximation to $x(1) = 99$.

Step size s	Approximation to $x(1)$
0.100000	53.355933
0.050000	75.881773
0.025000	91.639594
0.012500	97.671604
0.006250	98.856123
0.003125	98.988718
0.001563	98.999238
0.000781	98.999951
0.000391	98.999997

TABLE 2.6. Fourth order Runge-Kutta Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.

EXERCISES

EXERCISE 2.25. *The following formulas constitute the fourth order Runge-Kutta Algorithm :*

$$x_0 = x_0, \qquad x_{i+1} = \ x_i + s\tfrac{L_1+2L_2+2L_3+L_4}{6} \ \textit{for} \quad i = 0, 1, 2, \cdots$$

where

$$\begin{aligned} L_1 &= f(t_i, x_i) \\ L_2 &= f\left(t_i + \frac{s}{2}, x_i + \frac{s}{2}L_1\right) \\ L_3 &= f\left(t_i + \frac{s}{2}, x_i + \frac{s}{2}L_2\right) \\ L_4 &= f(t_i + s, x_i + sL_3) \end{aligned}$$

At each step one must calculate, in order, the four numbers L_1, L_2, L_3, and L_4 before calculating x_{i+1}.

(a) Use the fourth order Runge-Kutta method to approximate the solution of $x' = x$, $x(0) = 1$ at $t = 1$. Start with step size $s = 0.2$ and calculate a sequence of approximations by repeated step size halving.

(b) Use the solution formula $x = e^t$ to calculate percent errors. Do the errors decrease at the expected rate?

(c) Compare the results in (a) and (b) with those of the Euler and Heun's Algorithms in Tables 1.2 and 1.4.

EXERCISE 2.26. *Let $x = x(t)$ denote the solution of the initial value problem $x' = x^3$, $x(0) = 0.6$. It turns out that $x(1) \approx 1.1338934190$.*

(a) Use the Euler Algorithm to obtain an approximation to $x(1)$ with step size $s = 0.1$. How many correct significant digits does this approximation have?

(b) Obtain Euler approximations by repeatedly halving the step size (starting at $s = 0.1$*). At which step size* s *is the Euler approximation first correct to* 2 *decimal places? To* 3 *decimal places?*

(c) Compute the absolute error at each step size, starting from $s = 0.1$ *and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?*

Exercise 2.27. *Repeat Exercise 2.26 using Heun's Algorithm.*

Exercise 2.28. *Repeat Exercise 2.26 using the Runge-Kutta Algorithm.*

Exercise 2.29. *Repeat Exercise 2.26 using any other algorithm available on your computer.*

Exercise 2.30. *Let* $x = x(t)$ *denote the solution of the initial value problem* $x' = e^x$, $x(0) = 0$. *It turns out that* $x(0.8) \approx 1.6094379124$.

(a) Use the Euler Algorithm to obtain an approximation to $x(0.8)$ *with step size* $s = 0.1$. *How many correct significant digits does this approximation have?*

(b) Obtain Euler approximations by repeatedly halving the step size. At which step size s *is the Euler approximation first correct to* 2 *decimal places? To* 3 *decimal places?*

(c) Compute the absolute error at each step size, starting from $s = 0.1$ *and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?*

Exercise 2.31. *Repeat Exercise 2.30 using Heun's Algorithm.*

Exercise 2.32. *Repeat Exercise 2.30 using the Runge-Kutta Algorithm.*

Exercise 2.33. *Repeat Exercise 2.30 using any other algorithm available on your computer.*

Exercise 2.34. *Euler's Algorithm was derived by using the Rectangle Rule to approximate the integral* $\int_{t_i}^{t_{i+1}} f(t, x(t))dt$ *and Heun's Algorithm was derived by using the Trapezoid Rule. In this exercise you derive an algorithm by using Simpson's rule to approximate this integral. Simpson's rule for approximating an integral* $\int_a^b g(t)dt$ *is*

$$\int_a^b g(t)dt \approx \frac{1}{3}\left[g(b) + 4g\left(\frac{a+b}{2}\right) + g(a)\right].$$

(a) Use Simpson's rule to obtain a predictor-corrector algorithm for an approximation x_{i+1} *of the solution* $x = x(t)$ *of the initial value problem* $x' = f(t, x)$, $x(t_0) = x_0$ *at* $t = t_{i+1}$. *Use equal step sizes of length* s.

(b) Why is the algorithm derived in (a) called a "two step" algorithm? What problem does this cause at the start (i.e., at t_1*) and how might this be solved?*

(c) Use you answers from (a) and (b) to obtain an approximation to the solution of $x' = x$, $x(0) = 1$ *at* $T = 0.2$ *using a step size of* $s = 0.1$. *(The exact solution is* $e^{0.2} = 1.2214027582$*).*

(d) Compare your answer in (c) to the approximations obtained by the Euler and Heun's. If your computer has the fourth order Runge-Kutta Algorithm, compare its approximations also. Which algorithm gives the best approximation at $T = 0.2$*?*

Exercise 2.35. *Approximate the solution of the initial value problem* $x' = t^2 + x^2$, $x(0) = 0$ *at* $T = 0.5$ *using the Euler Algorithm, Heun's Algorithm, and the*

Runge-Kutta Algorithm. Start with step size $s = 0.1$ *and repeat by halving the step size four times. What are the accurate digits obtained from each algorithm? What is the best approximation obtained from all methods?*

EXERCISE 2.36. *Use a computer obtain an accurate graphical solution of the initial value problem* $x' = t^2 + x^2$, $x(0) = 0$ *on the interval from* $t = 0$ *to* $T = 1$ *using the Euler Algorithm. Repeatedly halve the step size* s *starting with* $s = 0.1$. *What step size did you stop with and why?*

EXERCISE 2.37. *Repeat Exercise 2.36 using Heun's Algorithm.*

EXERCISE 2.38. *Repeat Exercise 2.36 using the Runge-Kutta Algorithm.*

EXERCISE 2.39. *Repeat Exercise 2.36 using any other algorithm available on your computer.*

EXERCISE 2.40. *Use a computer obtain an accurate graphical solution of the initial value problem* $x' = \frac{x^3}{x-t}$, $x(0) = 1$ *on the interval from* $t = 0$ *to* $T = 1$ *using the Euler Algorithm. Use a window size of* $-20 < x < 20$. *Repeatedly decrease the step size* s *by a factor of one tenth, starting with* $s = 0.1$. *What step size did you stop with and why?*

EXERCISE 2.41. *Repeat Exercise 2.40 using Heun's Algorithm.*

EXERCISE 2.42. *Repeat Exercise 2.40 using the Runge-Kutta Algorithm.*

EXERCISE 2.43. *Repeat Exercise 2.40 using any other algorithm available on your computer.*

EXERCISE 2.44. *(a) Use any algorithm you wish to obtain a graphical solution of the initial value problem* $x' = 500\cos(200t)$, $x(0) = 0$. *Start with step size* $s = 0.1$ *and decrease until the graph has stabilized. What do you conclude about the solution?*

(b) Obtain a formula for the solution and use it to explain the graphical solution.

3. Chapter Summary & Exercises

A solution $x = x(t)$ of the differential equation $x' = f(t, x)$ is a differentiable function for which $x'(t) = f(t, x(t))$ holds for all t on an interval. In general a differential equation has infinitely many solutions. The general solution is the set of all solutions. We need an additional requirement in order to specify a unique solution. For a given point (t_0, x_0), the initial condition $x(t_0) = x_0$ is such a requirement. Theorem 1.1 gives conditions under which an initial value problem $x' = f(t, x)$, $x(t_0) = x_0$ has one and only one solution. Specifically, if $f(t, x)$ and its derivative $\partial f(t, x)/\partial x$ with respect to x are both continuous for t near t_0 and x near x_0, then there is one and only one solution. Although formulas for the solution cannot always be calculated, many kinds of approximation methods are available. The slope field associated with the differential equation helps in to sketching a graph of the solutions. A computers is useful for plotting the slope fields by the grid method; this method associates the slope $f(t, x)$ with each point (t, x) on from a chosen grid of points in the (t, x)-plane. Also useful for sketching slope fields are isoclines, which are curves in the (t, x)-plane made up of those points associated with a common slope. Numerical approximations to solution values $x(t)$ yield more accurate graphs of the solution. If x_1, $x_2, ...,$ x_n approximate the solution values $x(t_1)$, $x(t_2), ..., x(t_n)$ for $t_1 < t_2 < \cdots < t_n$, then by connecting the points (t_i, x_i) with straight line segments we construct an approximate (broken line segment) solution graph. Usually equally spaced points t_i are chosen and the common distance between them is the step size s of the method. If the approximations converge to the solution values as s tends to 0, then the broken line graph tends to the solution graph as s tends to 0. The Euler Algorithm is one method for calculating such approximations. It is based on the left hand rectangle rule for approximating an integral. Under the conditions on $f(t, x)$ in Theorem 1.1 the Euler approximations converge to the solution values as the step size s decreases to 0. The Euler Algorithm is of order 1, which means the errors tend to 0 at the same rate that s tends to 0. Faster converging algorithms are available. Heun's Algorithm is of order 2, which means the error tends to 0 at the same rate that s^2 tends to 0. A fourth order method called the Runge-Kutta Algorithm is commonly used.

EXERCISES

Find formulas for the general solutions of the differential equations below.

Exercise 3.1. $x' = \frac{1}{(1-t)t}$

Exercise 3.2. $x' = \frac{2t}{1+t^2}$

Find solution formulas for the following initial value problems.

Exercise 3.3. $x' = \frac{1}{1+t^2}$, $x(1) = \frac{\pi}{2}$

Exercise 3.4. $x' = \frac{1+t+t^2}{(1+t^2)t}$, $x(1) = 1$

Exercise 3.5. *Does existence and uniqueness Theorem 1.1 apply to the initial value problem* $x' = \sqrt{1-x}$, $x(1) = 0$*? Explain your answer. What do you conclude?*

Exercise 3.6. *Does the existence and uniqueness Theorem 1.1 apply to the initial value problem* $x' = \left(4 - x^2\right)^{-1}$, $x(2) = 0$. *Explain your answer. What do you conclude?*

For which initial values t_0 and x_0 does the existence and uniqueness Theorem 1.1 apply to the problems below? Explain your answer. What do you conclude? What can you conclude about initial value problems for other t_0 and x_0?

Exercise 3.7. $x' = \ln|x - t|$, $x(t_0) = x_0$

Exercise 3.8. $x' = \sqrt{9 - x^2 - t^2}$, $x(t_0) = x_0$

Exercise 3.9. $x' = |x|$, $x(t_0) = x_0$

Exercise 3.10. $x' = t^{\frac{1}{3}}x$, $x(t_0) = x_0$

Explain why Theorem 1.1 does not apply to the initial value problems below. What do you conclude?

Exercise 3.11. $x' = \sqrt{x^2 + t^2}$, $x(0) = 0$

Exercise 3.12. $x' = \sqrt{\sin(x^2 + t^2)}$, $x(0) = 0$

Exercise 3.13. *Apply the existence and uniqueness Theorem 1.1 to the initial value problem* $x' = \sqrt{1 - x^2}$, $x(0) = 0$ *in Example 1.5. What do you conclude?*

Exercise 3.14. *Let* $f(t, x)$ *be a polynomial in* t *and* x. *Prove that any initial value problem* $x' = f(t, x)$, $x(t_0) = x_0$ *has a unique solution on an interval containing* t_0.

Exercise 3.15. *Let* $p(z, w)$ *be a polynomial in* z *and* w *and let* $f(t, x) = p(\sin t, \sin x)$. *Prove that any initial value problem* $x' = f(t, x)$, $x(t_0) = x_0$, *has a unique solution on an interval containing* t_0.

Use a computer to obtain sketches of the slope fields associated with the following differential equations. By hand, sketch graphs of the solutions satisfying each of the given initial conditions.

Exercise 3.16. $x' = t^2 + 4x^2$
$x(0) = 0, \quad x(0.5) = 0.5$

Exercise 3.17. $x' = -\frac{t}{x}$
$x(0) = 1, \quad x(1) = -1$

Exercise 3.18. $x' = \frac{t^2 - x^2}{t^2 + x^2}$
$x(0) = 1, \quad x(-1) = -1$

Exercise 3.19. $x' = \ln\left(t^2 + x^2\right)$
$x(1) = 0, \quad x(0) = 0.1$

Exercise 3.20. *Find the isocline equation for the differential equations in Exercises 3.16-3.19 and graph several typical isoclines. Use your results to sketch the slope field of the equation.*

Use a computer to obtain sketches the slope fields associated with the equations in the initial value problems below. Hand sketch a graph of the solution satisfying each of the given initial conditions.

Exercise 3.21. $x' = 1 - x$
$x(0) = 0, \qquad x(0) = 1.5$

Exercise 3.22. $x' = x - 1$
$x(0) = 0, \qquad x(0) = 1.5$

Exercise 3.23. $x' = 1 - x^2$
$x(0) = 0, \qquad x(0) = 1.5$

Exercise 3.24. $x' = \sin(x^2 + t^2)$
$x(0) = 0, \qquad x(0) = -0.5$

Exercise 3.25. *Consider the initial value problem* $x' = x^3 e^{-t}$, $x(0) = 1$. *Apply the Euler Algorithm to approximate the solution at* $T = 0.6$.

(a) Start with step size $s = 0.1$ *and halve it four times. Which digits in the resulting approximations do you think are accurate? Explain your answer.*

(b) Halve the step size four more times. Now which digits in the resulting approximations do you think are accurate? Explain your answer.

Exercise 3.26. *Consider the initial value problem* $x' = x^3 e^{-t}$, $x(0) = 1$. *Apply Heun's Algorithm to approximate the solution at* $T = 0.6$.

(a) Start with step size $s = 0.1$ *and halve it four times. Which digits in the resulting four approximations do you think are accurate? Explain your answer.*

(b) Halve the step size four more times. Now which digits in the resulting four approximations do you think are accurate? Explain your answer.

Exercise 3.27. *Consider the initial value problem* $x' = x^3 e^{-t}$, $x(0) = 1$. *Apply the Runge-Kutta algorithm to approximate the solution at* $T = 0.6$. *(See Exercise 2.25.)*

(a) Start with step size $s = 0.1$ *and halve it four times. Which digits in the resulting four approximations do you think are accurate? Explain your answer.*

(b) Halve the step size four more times. Now which digits in the resulting four approximations do you think are accurate? Explain your answer.

Exercise 3.28. *Use the formula* $x(t) = (2e^{-t} - 1)^{-1/2}$ *for the solution of the initial value problem in Exercises 3.25, 3.26, and 3.27 to calculate the error and the per cent error of the approximations in these exercises for step size* $s = 0.00625$. *Round all numbers to* 6 *significant digits.*

EXERCISE 3.29. *Use the Euler Algorithm and a computer program to obtain an accurate graph of the solution of the initial value problem* $x' = 1.5x^3 \sin 10t$, $x(0) = 1$ *on the interval from* $t = 0$ *to* $T = 1$. *Use a window size of* $-2 < x < 2$. *Repeatedly halve the step size* s *starting with* $s = 0.2$. *At what step size did you stop and why?*

EXERCISE 3.30. *Repeat Exercise 3.29 using Heun's Algorithm.*

EXERCISE 3.31. *Repeat Exercise 3.29 using the Runge-Kutta Algorithm.*

EXERCISE 3.32. *Suppose the decay rate of a radioactive isotope is* $r = -0.35$ *per year. The differential equation for the amount* $x(t)$ *at time* t *is* $x' = -0.35x$.

(a) Use a computer to study the graphs of solutions with many different initial conditions $x_0 > 0$ *and formulate a conjecture about the length of time it takes a sample amount of the isotope to decay to one half of its initial amount.*

(b) Use the solution formula $x(t) = x_0 e^{-0.35t}$ *to verify or disprove your conjecture.*

EXERCISE 3.33. *Let* $x = x(t)$ *be the dollars in an investment account which is compounded continuously at a rate of* 4.5%.

(a) Perform numerical experiments on the model equations $x' = 0.045x$, $x(0) = s$ *to formulate a conjecture about how long will it take for the initial investment of* s *dollars to triple.*

(b) Use the solution formula $x(t) = te^{0.045t}$ *to prove or disprove your conjecture.*

EXERCISE 3.34. *Suppose a population has a per capita death rate* $d > 0$ *and a per capita birth rate that is proportional to population size* x *(with constant of proportionality denoted by* $a > 0$*).*

(a) Use the inflow-outflow rule (2.1) to write down a model differential equation for the population size $x = x(t)$.

(b) Perform numerical experiments and formulate a conjecture about the fate of the population. (Hint: choose a pair of model parameter values, such as $a = 1$ *and* $d = 1$, *and compute solution graphs for many initial population sizes* $x(0) = x_0$. *Then repeat for other values for* a *and* d.*)*

(c) Use the solution formula

$$x(t) = \frac{dx_0}{x_0 a + e^{dt}\left(d - x_0 a\right)}$$

to verify or disprove your conjectures in (b).

4. Applications

4.1. Bacterial Cell Growth. When placed in an environment of abundant resources (nutrients, space, etc.) cell cultures typically grow in such a way that their per capita rate of change is constant. Mathematically, this means the number of cells $x = x(t)$ at time t satisfies the differential equation

$$x' = rx$$

where the constant $r > 0$ is the "per capita growth rate". Often a particular microorganism's growth rate is described by the time it takes the number of cells in the culture to double. This time δ is called the "doubling time" (or "generation time") and it is related to the growth rate according to the formula

$$r = \frac{\ln 2}{\delta}.$$

For more detailed discussion of these topics and of population growth models see Sec.6, Chapter 3.

As an example, the doubling time of the bacterium *Staphylococcus aureus* is approximately $\delta = 30$ minutes, which corresponds to a per capita growth rate of

$$r = \frac{\ln 2}{30} = 0.02310 \text{ (per minute)}.$$

The growth of a culture of *S. aureus* initially consisting of 10^6 cells is described by the initial value problem

$$\begin{aligned} x' &= 0.02310x \\ x(0) &= 1. \end{aligned} \tag{4.1}$$

Here x is measured in units of 10^6 cells.

According to Theorem 1.1, this initial value problem has a unique solution $x = x(t)$. A slope field and a solution graph (drawn using Heun's Algorithm with step size $s = 0.05$) appear in Fig. 2.9. Notice the number of cells grows rapidly, following a seemingly exponential-like curve. Indeed, the solution formula for the initial value problem

$$x = e^{0.02310t}$$

shows the growth is indeed exponential.

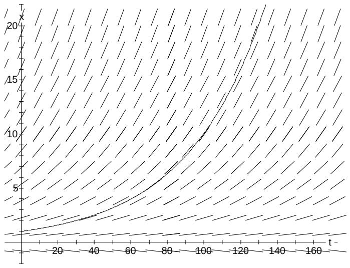

FIGURE 2.9. The slope field of the differential equation $x' = 0.02310x$ and the solution of the initial value problem (4.1) drawn using Heun's Algorithm with step size $s = 0.05$.

S. aureus is a common cause of bacterial skin infection (particularly in patients with HIV). The rapid exponential growth of a staph infection can be a serious problem if left untreated. Our modeling application involves determining the effect of a medical treatment that removes staph cells from the patient at a certain rate $h > 0$ (cells/minute). We set up our mathematical model (i.e., perform the Model Derivation Step in Fig. 2.1 of the Introduction) by applying the inflow-outflow rule (2.1) to the staph cell population numbers. This leads to the differential equation

$$x' = 0.02310x - h \tag{4.2}$$

More specifically, suppose a milligram (mg) of antibiotic in a particular patients kills staph cells at a rate of 10^4 per minute. Then a dosage of d mg kills a total of $10^4 d$ staph cells per minute. In units of 10^6 cells, we have

$$h = \frac{10^4}{10^6} d = 0.01d.\text{(per minute)} \tag{4.3}$$

Suppose, for the moment, that this removal rate h remains constant in time, as might be the case for example if the antibiotic were continuously administered intravenously. We want to know what dosages d, if any, will eliminate the staph infection from the patient, and if so in what amount of time.

The antibiotic kill rate h in (4.3) leads to the initial value problem

$$\begin{aligned} x' &= 0.02310x - 0.01d \\ x(0) &= 1. \end{aligned} \tag{4.4}$$

for the number of staph cell $x = x(t)$. Our next goal is to perform the Model Solution Step in the Modeling Cycle. What we want to learn from the solution $x = x(t)$ is whether or not it continues to increase or whether it decreases and eventually equals 0. The answer will presumably depend on the dosage d.

One way to obtain answers to our questions would be from a formula for the solution $x(t)$. We will learn how to find such a formula in Chapter 2. Here, however, we will investigate the solution by means of the methods developed in Sec. 2 and 2.2.

Fig. 2.10 shows slope fields and solution graphs, for a selection of dosages d, obtained by a computer. These graphs indicate the existence of a critical dosage level d_{cr} above which the staph infection is eliminated and below which it is not. From Fig. 2.10 this critical dose lies between 1.5 gm and 3.0 gm. Further computer explorations, using other values of d, suggest this critical value is approximately $d_{cr} = 2.31$ gm.

Another way to determine the critical value is to reason as follows. For $d < d_{cr}$, the staph infection increases ($x' > 0$) and for $d > d_{cr}$ it decreases ($x' < 0$). Therefore, at the critical dose $d = d_{cr}$ the infection should do neither, but instead remain constant. From the initial value problem (4.4), we see that x remains at $x(0) = 1$, and hence $x' = 0$, means

$$0.02310 - 0.01d_{cr} = 0$$

or

$$d_{cr} = 2.31.$$

At the critical dose d_{cr} the staph infection remains constant, but at a higher dose $d > d_{cr}$ our computer studies indicate that $x(t) = 0$ at some finite time t_c. This ("cured") time $t_c = t_c(d)$ when the infection is eliminated depends on d, as Fig. 2.10 shows. The higher the dose, the quicker the staph is eliminated; that is, $t_c(d)$ is a decreasing function.

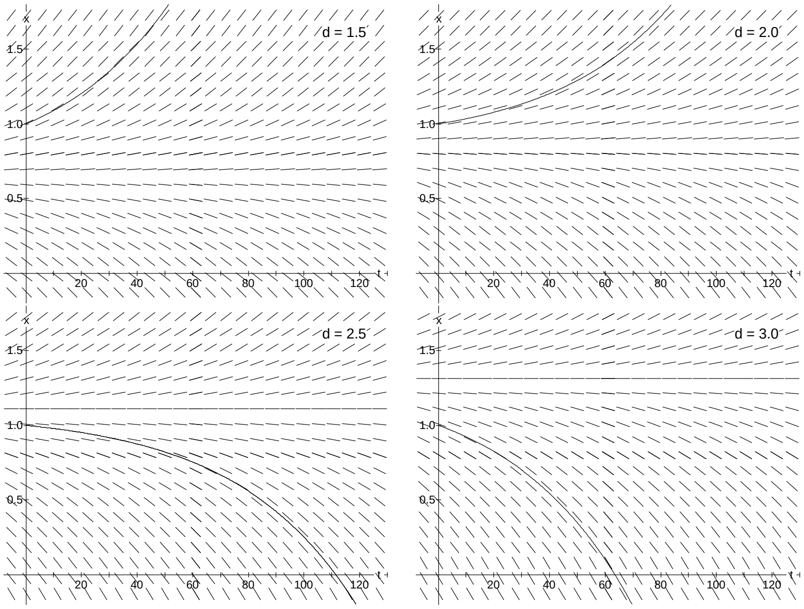

Figure 2.10. The slope field of the differential equation $x' = 0.02310x - 0.01d$ and the solution of the initial value problem (4.1) for selected values of the antibiotic dose d.

We emphasize that computer explorations do not “prove” our conclusions about the existence of a critical dosage and the dependence of t_c on d. This is because, when doing computer studies, we can calculate only a finite number of solutions for only a finite selection of dosages d. An advantage of a solution formula, if available (or, if not, other methods of analysis) is that these conclusions can be rigorously established. (See Exercise 6.36 in Chapter 2).

Often antibiotics are not continuously administered to a patient, but a dose is applied by pill or injection. In this case, the effect of the antibiotic is not constant, but decreases over time. To account for this change we return to the model equation (4.2) to see what adjustments must be made (this is the Model Modification Step of the Modeling Cycle). To proceed we need information concerning how the effectiveness of the antibiotic changes over time, so that we can derive a formula for the staph removal rate h.

Suppose, for example, the effectiveness of the antibiotic decreases exponentially so that

$$h = 0.01de^{-at}$$

Under this model assumption, the initial effectiveness of the antibiotic is $0.01d$ (cells/minute), but the effectiveness decreases over time with an exponential decay rate of $a > 0$. Suppose it is observed that the effectiveness decreases by 50% every hour. This allows us to calculate a. In 60 minutes, h is decreased by a fraction of 1/2 and therefore

$$e^{-a60} = 0.5.$$

or

$$a = 0.01155.$$

These assumptions lead us to a new initial problem for a staph infection starting with 10^6 cells:

$$\begin{aligned} x' &= 0.02310x - 0.01de^{-0.01155t} \\ x(0) &= 1. \end{aligned} \tag{4.5}$$

(Recall x is measured in units of 10^6.)

Again we ask: what dosages d, if any, will eliminate the staph infection?

Fig. 2.11 shows the slope field and the solution of the initial value problem (4.5) for some selected values of the dose d. These samples suggest that this initial value problem also has a critical dosage d_{cr} below which the treatment does not eliminate the staph infection. The particular examples in Fig. 2.11 indicate that d_{cr} lies between 2.5 gm and 4.0 gm. (See Exercise 6.37 in Chapter 2.)

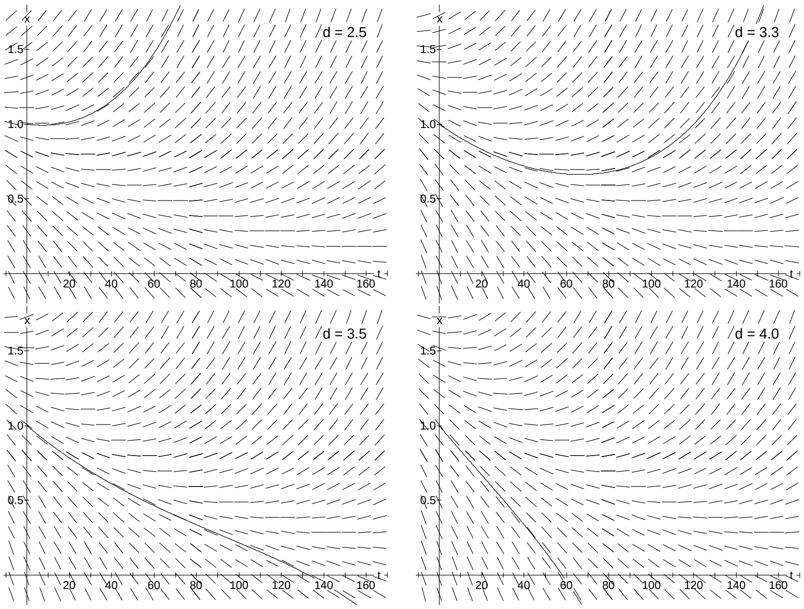

FIGURE 2.11. The slope field of the differential equation $x' = 0.02310x - 0.01de^{-0.01155t}$ and the solution of the initial value problem (4.5) for selected values of the antibiotic dose d.

An interesting difference between the intravenous treatment modeled by (4.4) and the pill or injection treatment modeled by (4.5) occurs for doses below the critical level d_{cr}. Unlike the intravenous treatment, the pill or injection treatment can show an initial improvement (x initially decreases in Fig. 2.11 for $d = 2.5$ and 3.3) even though the infection ultimately "bounces back" and grows unabated. Thus, one must guard against a mistaken conclusion, based on its early effectiveness, that the treatment will result in a cure.

4.2. Running a Curve. One of the most famous laws of physics is Newton's law of motion given by the equation $F = ma$. Here m is the mass of a moving object and a is its acceleration. The letter F represents the force (or a collection of many forces $F = F_1 + F_2 + \cdots$) acting on the object. Since

$$a = \frac{dv}{dt} = \frac{d^2x}{dt},$$

where v is the object's velocity and x is its position (measured from some reference point), the application of Newton's law usually results in a differential equation that describes the motion of the object when subject to the force F. The Modeling Derivation step of the Modeling Cycle involves describing the relevant forces acting on the object so as to obtain a mathematical expression for F. We will utilize this law in a variety of applications throughout the book. In this section, we will use Newton's law to study a sprinter running a race of fixed distance.

One model of a sprinter running in a straight line assumes two forces are involved: the propulsive force exerted by the runner and a resistive force (due mainly to air resistance).[4] Thus, $F = F_p + F_r$ in Newton's law. In this model it is assumed that the runner exerts a constant propulsive force throughout the race, an assumption that seems reasonable for sprint of short distance. Thus,

$$F_p = mf$$

were $f > 0$ is the per unit mass force characteristic of a particular individual runner. The resistive force, on the other hand, depends on the runner's velocity. It is absence when the runner is not moving and it increases with the runner's speed. The simplest law assumes the resistive force is proportional to velocity v, i.e.,

$$F_r = -cv$$

where the "coefficient of friction" $c > 0$ is another characteristic of each particular runner. The reason for the negative sign is that the resistive force works against the runner.

In the absence of other forces, Newton's law yields the differential equation

$$m\frac{dv}{dt} = mf - cv$$

for the runner's velocity v. If we divide both sides by m and denote the per unit mass coefficient of friction c/m by σ, this equation becomes

$$\frac{dv}{dt} = f - \sigma v. \tag{4.6}$$

The model parameters f and σ can be approximated from performance records. For example, the parameters for 1968 Mexico City Olympics gold medalist Tommie Smith have been estimated to be $f = 13.46$ (Newtons,/kg) and $\sigma = 1.252$ (per second).[5]

In races one is usually interested in the time it takes to run a fixed distance x_d from a starting line at $x = 0$ from which the runner's begin from a standing start ($v(0) = 0$). To determine this time from the initial value problem

$$\frac{dv}{dt} = f - \sigma v, \quad v(0) = 0$$

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[4] J. B. Keller, 1973. *Physics Today*, 26(9), p.42

[5] A. Armenti, Jr., 1993. *The Physics of Sports*, American Institute of Physics, New York, pp. 105-108

we calculate the runner's position from $v = dx/dt$, i.e.,

$$x(t) = \int_0^t v(s)ds,$$

and then solve the equation $x(t) = x_d$ for t.

For example, consider gold medalist Tommie Smith running a 100m race. We can use a computer to approximate the solution of the initial value problem

$$\begin{aligned} \frac{dv}{dt} &= 13.46 - 1.252v \\ v(0) &= 0. \end{aligned} \tag{4.7}$$

The algorithms studied in Sec. 2.2 produce approximations to the velocity $v(t)$ at points t_i (depending on the chosen step size s) lying in, say, the interval $0 \leq t \leq 12$. The resulting table of approximations for $v(t_i)$ permits us to approximate the distance

$$x(t_i) = \int_0^{t_i} v(s)dx$$

run at each point in time t_i by using an numerical integration procedure (for example, the trapezoid rule). Table 2.7 shows some of the results.

We make two observations from the numerical solution of the initial value problem (4.7). First, the model predicts that Tommie Smith could run 100m in approximately 10.1 seconds (Table 2.7). Secondly, the model predicts that after about 6 seconds (55 m), Smith's velocity $v(t)$ is very nearly constant at 10.75 (m/sec) for the rest of the race.

t	$x(t)$	$v(t)$
10.00	98.92	10.75
10.01	99.03	10.75
10.02	99.14	10.75
10.03	99.24	10.75
10.04	99.35	10.75
10.05	99.46	10.75
10.06	99.57	10.75
10.07	99.67	10.75
10.08	99.78	10.75
10.09	99.89	10.75
10.10	99.99	10.75
10.11	100.10	10.75
10.12	100.21	10.75

TABLE 2.7. Some results of applying Heun's Algorithm (Sec. 2.3) with step size $s = 0.01$ to the initial value problem (4.7).

Some sprints are not run in a straight line, but involve running a curve at the beginning of the race, with staggered starting positions for the racers. For example, this is the case for most 200m races in which the course lies on a (circular) curve for 100m before straightening out for the last 100m.

When running along the curve, the sprinter's propulsive force must supply an additional centripetal acceleration which depends on the radius of curvature of the curve. Furthermore, the radius of curvature is different for each lane. Lanes are typically 1.22 meters wide and the inner radius of the n^{th} lane is given by the formula

$$R(n) = \frac{100}{\pi} + 1.22\,(n-1)\,.$$

We will not delve into the physics of the derivation here, but leave it to say that the new equation motion that results when the additional force due to the centripetal acceleration is taken into account leads to the initial value problem[6]

$$\begin{aligned} \frac{dv}{dt} &= \left(f^2 - \left(\frac{v^2}{R(n)}\right)^2\right)^{1/2} - \sigma v \\ v(0) &= 0. \end{aligned} \tag{4.8}$$

These are the equations of motion during the first 100m of the 200m race.

During the second 100m of the race equation (4.6) is applicable. The initial condition associated with (4.6) would be the runner's velocity v_c at the end of the first 100m (along the curve portion) of the race.

Thus, the Model Solution Step of the Modeling Cycle involves, in this application, the numerical solution of the initial value problem (4.6) until the time t_c is reached at which $x(t_c) = 100$. At this time the runner's velocity is $v_c = v(t_c)$, which constitutes the initial condition for equation (4.8). This initial value problem is solved until the finishing time for the runner is reached, i.e., the time t_f at which $x\left(t_f\right) = 100$ (the second 100m of the race).

We can, however, simply the second step of the solution procedure as follows. It will turn out (as in the 100m sprint example above) that the runner's velocity will reach a constant by the time the final 100m portion of the race is reached. Therefore, rather than solve a second initial value problem using equation (4.8) we can obtain a good approximation to the time for the last 100m by assuming a constant velocity v_f is maintained, in which case the final 100m time is given by the formula $100/v_c$. The model predicted sprint time for the 200m race is then

$$t_f = t_c + \frac{100}{v_c}. \tag{4.9}$$

Notice all that is needed by the model to make a prediction for a sprinter's time in a 200m race are the parameter values f and σ (obtained from the sprinter's performance data on straight courses) and the lane assignment n.

As an example we consider gold medalist Tommie Smith's performance in the 1968 Mexico City Olympics. In the 200m finals Smith was assigned lane $n = 3$. Using $f = 13.46$, $\sigma = 1.252$ and $R(3) = 34.27$ we approximate the solution of (4.8) using Heun's Algorithm.

From Table 2.8 we see that the model predicts Smith will run the first 100 meters along the curve in approximately $t_c = 10.33$ seconds and at the end of the curve his velocity will be approximately $v_c = 10.45$. From (4.9) we calculate the predicted time for Smith's 200m sprint in lane $n = 3$ to be approximately

$$t_f = 10.33 + \frac{100}{10.45} = 19.90.$$

In fact Smith ran the race in 19.83 seconds (at that time a world record).

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[6] A. Armenti, Jr., 1993. *The Physics of Sports*, American Institute of Physics, New York, pp. 105-108

t	$x(t)$	$v(t)$
10.25	99.20	10.45
10.26	99.31	10.45
10.27	99.41	10.45
10.28	99.51	10.45
10.29	99.62	10.45
10.30	99.72	10.45
10.31	99.83	10.45
10.32	99.93	10.45
10.33	100.0	10.45
10.34	100.1	10.45
10.35	100.2	10.45
10.36	100.4	10.45
10.37	100.6	10.45

TABLE 2.8. Some results of applying Heun's Algorithm with step size $s = 0.01$ (Sec. 2.3) with step size $s = 0.01$ to the initial value problem (4.8) with $f = 13.46$, $\sigma = 1.252$ and $R(3) = 34.27$.

We can use the model (4.8) to predict what might have been the result if Smith been given a different lane assignment. For example, the results in Table 2.9 for lane $n = 1$ show a slower predicted time of

$$t_f = 10.36 + \frac{100}{10.40} = 19.98.$$

Runners dislike lane 1 as being "too tight". The slower time predicted by the model for $n = 1$ bears out this opinion. Had Smith run in lane $n = 8$, however, his world record, according to the model, would have been even lower than 19.83 seconds. See Exercise 4.10.

t	$x(t)$	$v(t)$
10.25	98.85	10.40
10.26	98.96	10.40
10.27	99.06	10.40
10.28	99.17	10.40
10.29	99.27	10.40
10.30	99.37	10.40
10.31	99.48	10.40
10.32	99.58	10.40
10.33	99.69	10.40
10.34	99.79	10.40
10.35	99.89	10.40
10.36	100.0	10.40
10.37	100.1	10.40

TABLE 2.9. Some results of applying Heun's Algorithm with step size $s = 0.01$ (Sec. 2.3) with step size $s = 0.01$ to the initial value problem (4.8) with $f = 13.46$, $\sigma = 1.252$ and $R(1) = 31.83$.

EXERCISES

When we talk of a population's doubling time in the exercises below we imply that the population grows, in the absence of any limiting facts, with a constant per unit growth rate: $x' = rx$.

EXERCISE 4.1. *E. coli has a doubling time of approximately 20 minutes. Assume* h *cells per minute are removed from a culture initially at* 10^8 *cells. Use a computer to solve the initial value problem for the number of cells* $x = x(t)$ *at time* t*. Determine the critical value* h_{cr} *of* h *above which the culture will die out.*

EXERCISE 4.2. *E. coli has a doubling time of approximately 20 minutes. Assume* he^{-at} *cells per minute are removed from a culture initially at* 10^8 *cells. Use a computer to solve the initial value problem for the number of cells* $x = x(t)$ *at time* t*. Explore those values of* a *and* h *for which the culture goes extinct. Specifically, for selected values of* a*, calculate the critical value* h_{cr} *for* h *above which the culture goes extinct. Determine a relationship between* a *and* h_{cr}*.*

EXERCISE 4.3. *The bacterium M. Tuberculosis has a doubling time of approximately 13 hours. Assume* h *cells per hour are removed from a culture initially at* 10^7 *cells. Use a computer to solve the initial value problem for the number of cells* $x = x(t)$ *at time* t*. Determine the critical value* h_{cr} *of* h *above which the culture will die out.*

EXERCISE 4.4. *The bacterium M. Tuberculosis has a doubling time of approximately 13 hours. Assume* he^{-at} *cells per hour are removed from a culture initially at* 10^7 *cells. Use a computer to solve the initial value problem for the number of cells* $x = x(t)$ *at time* t*. Explore those values of* a *and* h *for which the culture goes extinct. Specifically, for selected values of* a*, calculate the critical value* h_{cr} *for* h *above which the culture goes extinct. Determine a relationship between* a *and* h_{cr}*.*

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EXERCISE 4.5. *In the model (4.5) suppose the effectiveness of the antibiotic decays more slowly than exponentially. Specifically, assume* $h = 200d\,(1 + at)^{-1}$ *where* $a > 0$ *is a constant. Assume there is a 50% drop in effectiveness after 60 minutes.*

(a) Modify the initial value problem (4.5) to account for this new assumption.

(b) Using slope fields and computer calculated solution graphs, determine whether or not this new model has a critical dosage value d_{cr} *below which the infection is not controlled and above which the infection is eliminated.*

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No population can grow exponentially indefinitely. Many populations eventually decrease their rate of growth and level off at a number K appropriate to its environment and available resources. A differential equation often used to model this kind of growth is $x' = rx\,(1 - x/K)$ where r is the exponential growth rate at low population numbers. Suppose such a population is harvested at a constant rate h. Then the equation governing the populations growth is $x' = rx\,(1 - x/K) - h$. As an example, suppose the number of fish in a large lake grows according to this law. It is estimated that the lake can support $K = 10^4$ fish, and it is known that during the exponential grow phase (i.e., low population numbers) the fish population will double in two years. Use a computer to investigate the following question: at what maximal annual rate h_{cr} can the fish be harvested without causing extinction if initially there are the following numbers in the lake?

Exercise 4.6. $x(0) = 10^2$

Exercise 4.7. $x(0) = 10^3$

Exercise 4.8. $x(0) = 10^4$

Exercise 4.9. *Investigate many initial conditions* $x(0) > K/2 = 0.5 \times 10^4$*. What do you notice about* h_{cr}*?*

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Exercise 4.10. *Calculate the model (4.8) predicted time for gold medalist Tommie Smith had he run in lane* $n = 8$ *of the 200m finals in the 1968 Mexico City Olympics.*

Exercise 4.11. *The current world record for 200m of 19.32 seconds, set at the 1996 Atlanta Olympics, is held by Michael Johnson. In order for Tommie Smith to equal this time on a straight course what higher value of the per unit mass propulsive form* f *would he have to attain?*

Exercise 4.12. *The current world record for 200m of 19.32 seconds, set at the 1996 Atlanta Olympics, is held by Michael Johnson. In order for Tommie Smith to better this time on a curved course in lane* $n = 3$ *what higher value of the per unit mass propulsive form* f *would he have to attain?*

Exercise 4.13. *Estimated parameter values for sprinter Jim Hines are* $f = 7.10$ *(N/kg) and* $\sigma = 0.581$ *(per second). In a 100m (straight line) race who would win, Jim Hines or Tommie Smith?*

Exercise 4.14. *Estimated parameter values for sprinter Jim Hines are* $f = 7.10$ *(N/kg) and* $\sigma = 0.581$ *(per second). In a 200m (curve course) race who would win, Jim Hines in lane* $n = 5$ *or Tommie Smith in lane* $n = 4$*?*

Exercise 4.15. *Who would win if Hines and Smith switched lanes in Exercise 4.14?*