MATH 355-002
Spring 2012

HAND-IN HOMEWORK #3
(due Thursday, February 2)

ANSWERS

Chapter 1.2 (p 49): Exercise 2.30. Let $x = x(t)$ denote the solution of the initial value problem $x' = e^x$, $x(0) = 0$. It turns out that $x(0.8) \approx 1.6094379124$.

(a) Use the Euler Algorithm to obtain an approximation to $x(0.8)$ with step size $s = 0.1$. How many correct significant digits does this approximation have?

(b) Obtain Euler approximations by repeatedly halving the step size. At which step size s is the Euler approximation first correct to 2 decimal places? To 3 decimal places?

(c) Compute the absolute error at each step size, starting from $s = 0.1$ and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?

ANSWER:

(a) $x(0.8) \approx 1.336261$, *which has one correct significant digit*

(b) $s = 0.001563$ *and* $s = 0.9765625 \times 10^{-4}$

(c)

Step size s	$x(1) \approx$ Euler	Absolute Error	% error
0.10000	1.336261	0.27318	16.97
0.05000	1.448299	0.16114	10.01
0.02500	1.520285	0.08915	5.54
0.01250	1.562228	0.04721	2.93
0.00625	1.585094	0.02434	1.51

The error goes down by approximately a fraction of $1/2$ *at each step, as is to be expected from the first order Euler Algorithm.*

Chapter 1.2 (p 49): Exercise 2.31. Repeat Exercise 2.30 using Heun's Algorithm.

ANSWER:

(a) $x(0.8) \approx 1.594088$, *which has one correct significant digit*

(b) $s = 0.05$ *and* $s = 0.0125$

(c)

Step size s	$x(1) \approx$ Heun	Absolute Error	% error
0.10000	1.594088	15.35×10^{-3}	9.54×10^{-1}
0.05000	1.605369	4.069×10^{-3}	2.53×10^{-1}
0.02500	1.608403	1.035×10^{-3}	0.64×10^{-1}
0.01250	1.609178	0.260×10^{-3}	0.16×10^{-1}
0.00625	1.609373	0.065×10^{-3}	0.04×10^{-1}

The error goes down by approximately a fraction of $1/4 = (1/2)^2$ *at each step, as is to be expected from the second order Heun's Algorithm.*

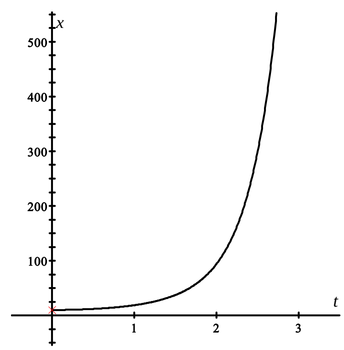
Chapter 1.2 (p 49): Exercise 2.32. Repeat Exercise 2.30 using the Runge-Kutta Algorithm.
ANSWER:
(a) $x(0.8) \approx 1.6094571436$*, which has five correct significant digits*
(b) $s = 0.1$ *and* $s = 0.1$
(c)

Step size s	$x(1) \approx$ Runge-Kutta	Absolute Error	% error
0.10000	1.6094571436	-1.9232×10^{-5}	1194.90×10^{-6}
0.05000	1.6094402828	-2.371×10^{-6}	147.28×10^{-6}
0.02500	1.6094381088	-1.97×10^{-7}	12.20×10^{-6}
0.01250	1.6094379264	-1.4×10^{-8}	0.87×10^{-6}
0.00625	1.6094379134	1.0×10^{-9}	0.06×10^{-6}

The error goes down by approximately a fraction of $1/16 = (1/2)^4$ *at each step, as is to be expected from the fourth order Runge-Kutta Algorithm.*

Chapter 2.1 (p 75): Exercise 1.24. Use a computer program to plot the graph of the solution of the initial value problem $x' = tx + e^t$, $x(0) = 10$. On what interval does the solution appear to exist? Apply Corollary 1.1. to determine on what interval the solution exists.

ANSWER: *From computer simulations (for example, see the accompanying plot) the solution might appear to have a vertical asymptote, but Corollary 1.1 implies that the solution exists on the entire real line since the coefficients* $p(t) = t$ *and* $q(t) = e^t$ *are continuous on the entire real line.*



Chapter 2.1 (p 76): Exercise 1.45. Find a formula for the general solution of the linear nonhomogeneous equation (a and b are constants):

$$x' = ax + \sin bt$$

ANSWER: *From the Variations of constants formula*

$$\begin{aligned} x(t) &= ce^{P(t)} + e^{P(t)} \int e^{-P(t)} q(t)\, dt \text{ where } P(t) = \int a dt = at \\ x(t) &= ce^{at} + e^{at} \int e^{-at} \sin bt dt \\ &= ce^{at} + e^{at} \left(-\frac{1}{a^2+b^2} e^{-at} (b \cos bt + a \sin bt) \right) \\ &= ce^{at} - \frac{1}{a^2+b^2} (b \cos bt + a \sin bt) \end{aligned}$$

provided $a^2 + b^2 \neq 0$. *If* $a^2 + b^2 = 0$, *i.e., if* $a = b = 0$, *then* $x' = 0$ *and the general solution is* $x(t) = c$.

Chapter 2.1 (p 77): Exercise 1.61. Find a formula for the solution of the initial value problem (a and b are constants):

$$x' = x\cos at + b\cos at, \quad x(0) = 0.$$

ANSWER: *From the Variations of constants formula*

$$x(t) = x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(s)} q(s)\, ds \text{ where } P(t) = \int_{t_0}^{t} p(s)\, ds.$$

If $a \neq 0$, *we have*

$$P(t) = \int_0^t \cos as ds = \frac{1}{a}\sin as\Big|_{s=0}^{s=t} = \frac{1}{a}\sin at$$

and

$$\begin{aligned} x(t) &= 0 + e^{\frac{1}{a}\sin at} \int_0^t e^{-\frac{1}{a}\sin as} (b\cos as)\, ds \\ x(t) &= e^{\frac{1}{a}\sin at} \left(-be^{-\frac{1}{a}\sin as}\right)\Big|_{s=0}^{s=t} \\ &= e^{\frac{1}{a}\sin at} \left(-be^{-\frac{1}{a}\sin at} - (-b)\right) \\ &= be^{\frac{1}{a}\sin at} - b \text{ if } a \neq 0. \end{aligned}$$

If $a = 0$, *we have the autonomous equation* $x' = x + b$ *for which* $x_h(t) = ce^t$ *and* $x_p = -b$. *Thus,* $x(t) = ce^t - b$ *and* $x(0) = c - b = 0$ *implies* $c = b$. *This gives the solution*

$$x(t) = be^t - b \text{ if } a = 0.$$

Chapter 2.3 (p 91): Exercises 3.12. For the equation

$$x' = 4x - 3e^{\pi t}$$

(a) construct the appropriate "guess" for a particular solution x_p and (b) use the "guess" to find a particular solution of the equation.

ANSWER:

(a) The general solution of the associated homogeneous equation $x' = 4x$ *is* $x_h(t) = ce^{4t}$. *The term* $q(t) = -3e^{\pi t}$ *is a multiple of the exponential* $e^{\pi t}$ *which yields no new and independent functions upon repeated differentiation. Therefore, we look for a particular solution in the form* $x_p(t) = ke^{\pi t}$.

(b) A substitution of $x_p(t) = ke^{\pi t}$ *into the nonhomogeneous differential equation yields*

$$\pi k e^{\pi t} = 4ke^{\pi t} - 3e^{\pi t} \text{ or } 0 = ((4-\pi)k - 3)e^{\pi t}$$

and hence $k = 3/(4-\pi)$. *A particular solution is*

$$x_p = \frac{3}{4-\pi} e^{\pi t}$$

and the general solution is

$$x = x_h + x_p = ce^{4t} + \frac{3}{4-\pi} e^{\pi t}.$$

Chapter 2.3 (p 91): Exercises 3.14. For the equation

$$x' = -2x - 3e^{-2t}$$

(a) construct the appropriate "guess" for a particular solution x_p and (b) use the "guess" to find a particular solution of the equation.

ANSWER:

(a) The general solution of the associated homogeneous equation $x' = -2x$ *is* $x_h(t) = ce^{-2t}$*. The term* $q(t) = -3e^{-2t}$ *is a multiple of the exponential* e^{-2t} *which is a solution of the homogeneous equation. Therefore, we multiple it by* t to obtain te^{-2t} which *yields only one new and independent function upon repeated differentiation, namely* e^{-2t}*. Therefore, we look for a particular solution in the form* $x_p(t) = kte^{-2t}$.

(b) A substitution of $x_p(t) = kte^{-2t}$ *into the nonhomogeneous differential equation yields*

$$\begin{aligned} -2kte^{-2t} + ke^{-2t} &= -2kte^{-2t} - 3e^{-2t} \\ (k+3)e^{-2t} &= 0 \end{aligned}$$

and hence $k = -3$*. A particular solution is*

$$x_p = -3te^{-2t}$$

and the general solution is

$$x = x_h + x_p = ce^{-2t} - 3te^{-2t}.$$