On a Conjecture of Levi-Civita

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1. Introduction. In 1925 Levi-Civita [5] reduced the mathematical theory of two-dimensional, periodic, permanant waves of finite amplitude on an ideal fluid of infinite depth to the problem of determining all non-constant complex valued functions $\omega(\zeta) = \theta + i\tau$ of a complex variable $\zeta = \rho e^{i\sigma} = \xi + i\eta$ holomorphic in the unit disk $|\zeta| \leqq 1$ which satisfy $\omega(0) = 0$ and

$$\partial\theta/\partial\rho = pe^{-3\tau} \sin \theta \quad \text{on} \quad |\zeta| = 1, \tag{1.1}$$

where p is an *a priori* undetermined positive constant. It is assumed that $\omega(\zeta) \,\varepsilon\, C'(|\zeta| \leqq 1)$ and without loss of generality that ω is to be purely imaginary for real ζ [5]. This problem will be referred to as problem (I).

Each such function ω for which $|\omega|$ is sufficiently small gives rise to a periodic wave (and conversely) where θ is the angle of inclination of the flow vector with the horizontal and τ is related to the speed q of the flow vector, *viz.*, $\tau = \log (q/c)$ where c is the speed of the wave profile. The constant p is related to physical constants by

$$p = \frac{g\lambda}{2\pi c^2}, \tag{1.2}$$

where λ is a wavelength of the wave and g is the acceleration due to gravity.

One is interested in the existence of waves in a neighborhood of the uniform flow $\omega \equiv 0$ and, consequently, requires that $|\omega|$ be small. Specifically it is required that

$$|e^{-i\omega} - 1| \leqq \beta < 1, \qquad |\zeta| \leqq 1, \tag{1.3}$$

for some constant β, $0 < \beta < 1$. Using the technique of analytic expansion in powers of a small parameter (essentially the amplitude of the wave), Levi-Civita demonstrates the existence of a non-constant solution ω to problem (I) satisfying (1.3) for any value of p satisfying $1 - \epsilon \leqq p \leqq 1$ where $\epsilon > 0$ is some sufficiently small constant. The result has been proved also by Littman and Nirenberg (see Stoker [9]) using a fixed point theorem and by Nekrasov [8] and Lichtenstein [6] using the theory of integral equations. Each such solution to problem (I) gives rise to a unique wave motion as soon as a wavelength λ is specified; c is then determined from (1.2).

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Levi-Civita remarks that if one is interested only in the existence of a solution to problem (I) one can restrict attention to $p \leqq 1$ since $\bar{\omega} = \omega(\zeta^n)$, where $n =$ any positive integer, is a solution corresponding to $\bar{p} = np$. This solution arises from considering a wave of wavelength λ (generated by ω) as a wave of wavelength $\bar{\lambda} = n\lambda$; consequently, the waves arising from $\bar{\omega}$ are physically identical to those arising from ω. Levi-Civita then asks whether solutions for which $p > 1$ other than those constructed from solutions for $p \leqq 1$ are possible and conjectures that such is not the case.

Treating this problem as a uniqueness question for harmonic functions under the non-linear boundary condition (1.1), several authors have attempted to prove this conjecture (*cf.* [2], [3], [7]), but have only succeeded in excluding the existence of two solutions to the problem whose hodograph vectors (q, θ) bear a complicated relation to one another (*cf.* [3] and [7]). It is not difficult to see, however, that a general uniqueness theorem for this problem is not possible since there exist values of $\bar{p} > 1$ (sufficiently large) which are integer multiples of more than one value of p in the range $1 - \epsilon \leqq p \leqq 1$ and, consequently, the problem for this value of $\bar{p}$ has more than one solution. The conjecture must be approached from another point of view. In §2 we discuss how the problem may be viewed in terms of the minimal wavelength of a wave and in §3 we prove Levi-Civita's conjecture for $p \geqq (1 + \beta)^3$ where β is the constant appearing in (1.3).

2. Preliminary considerations. The question Levi-Civita is asking is whether all waves of the type under consideration are given by solutions to problem (I) corresponding to $p \leqq 1$; if so, then no loss in generality occurs if this restriction is placed on p. This question may be formulated in terms of $p_m = g\lambda_m/2\pi c^2$ where λ_m is the minimal wavelength of a given wave (*i.e.*, the smallest positive real number which is a wavelength of the wave) as follows: *is $p_m \leqq 1$ for any given wave?* If such is the case, then all waves (whether obtained from solutions to problem (I) for $p > 1$ or not) may in fact be obtained from solutions to problem (I) for $p = p_m \leqq 1$, since any wave (as proved by Levi-Civita) gives rise to a solution to problem (I) for p as given by (1.2) for *any* wavelength λ. In §3 (*cf.* Theorem 3.1) we prove that $p_m < (1 + \beta)^3$ for any wave of the type under consideration excepting certain peculiar types (see below). Whether or not there actually exist waves for which $1 < p_m < (1 + \beta)^3$ remains an open question.

To formulate the existence problem for waves of the type under consideration as problem (I), the region of flow beneath a wave of length λ and bounded by vertical lines d', d'' passing through two nodes is mapped conformally onto the unit disk in such a way that d', d'' are mapped onto the split segment $-1 \leqq \operatorname{Re} \zeta < 0$, $\operatorname{Im} \zeta = 0$, on the real axis and the vertical line d''' passing through the crest occurring midway between the nodes is mapped onto $0 < \operatorname{Re} \zeta \leqq 1$, $\operatorname{Im} \zeta = 0$. In his paper Levi-Civita proves that the wave is necessarily symmetric with respect to d', d'' and d''' and consequently θ vanishes along these lines (hence the requirement in problem (I) that $\theta = 0$ for real ζ). Throughout this paper *we exclude from consideration waves for which* θ *vanishes at any point other than those lying on* d', d'' *and* d''' and as a result we may formulate the following criterion: *for a solution* $\omega = \theta + i\tau$ *to problem* (I) *corresponding to a given value of* p *we have* $\lambda_m < \lambda$ *for all* λ *if and only if* $\theta = 0$ *for some* ζ, $|\zeta| \leqq 1$, *for which* $\text{Im } \zeta \neq 0$. Although we have excluded waves of a special type for purely mathematical reasons, one suspects that stable waves of this type do not exist; a proof of this fact would certainly be of interest.

3. Results. We begin by proving two lemmas.

Lemma 3.1. *If* $\bar{\theta} = \bar{\theta}(\xi, \eta) \not\equiv$ *const. satisfies the equation*

$$\eta \Delta u + 2u_\eta = 0, \tag{3.1}$$

on a regular domain D [4] *and* $\bar{\theta} \in C^2(D)$, $\bar{\theta} \in C'(D + \partial D)$, *then the outer normal derivative* $\partial\bar{\theta}/\partial n$ *changes sign on the boundary* ∂D.

Proof. Applying the divergence theorem to $\eta^2 \partial\bar{\theta}/\partial n$ on ∂D, we have for any solution $\bar{\theta}$ to (3.1)

$$\int_{\partial D} \eta^2 \frac{\partial \bar{\theta}}{\partial n} \, ds = 0,$$

and the lemma follows, unless $\partial\bar{\theta}/\partial n \equiv 0$ on ∂D. But in the latter case an application of the divergence theorem to $\eta^2 \bar{\theta} \partial\bar{\theta}/\partial n$ readily proves $\bar{\theta} \equiv$ const.

Lemma 3.2. *Consider the function of polar coordinates* $x = r \cos \theta$, $y = r \sin \theta$ *given by* $f(r, \theta) = \sin \theta / r^3 \theta$ *on the circle* $C_\beta{:}(x - 1)^2 + y^2 \leqq \beta^2$, $0 < \beta < 1$. *Then for the principal values of* θ

$$\min_{C_\beta} f(r, \theta) = (1 + \beta)^{-3}.$$

Proof. Since $f_r < 0$ on C_β the minimum of f is assumed on ∂C_β and, moreover, on that portion of ∂C_β for which $r = \cos \theta + (\beta^2 - \sin^2 \theta)^{1/2}$, $-\sin^{-1} \beta \leqq \theta \leqq \sin^{-1} \beta$. A substitution of this expression for r into f and a straightforward calculation of $df/d\theta$ shows that the minimum of f on ∂C_β occurs at $\theta = 0$.

We now establish our main result.

Theorem 3.1. *For any wave which satisfies inequality* (1.3) (*and the restriction of* §2), *we have* $p_m < (1 + \beta)^3$.

Proof. Suppose, on the contrary, that there exists a wave for which $p_m \geqq (1 + \beta)^3$. Then there exists a non-constant solution $\omega = \theta + i\tau$ to problem (I) satisfying (1.3) for $p = p_m \geqq (1 + \beta)^3$ where, according to the criterion of §2, $\theta = 0$ if and only if $\eta = 0$. Since θ must change sign on ∂D (this may be seen mathematically from the well known integral theorem of Gauss [1] and (1.1)), θ possess opposite signs on the upper and lower half disk, *i.e.* in $D + \partial D$

$$(3.2)\qquad \begin{array}{llll} \theta > 0, & \eta > 0 & \text{and} & \theta < 0, \quad \eta < 0, \text{ or} \\ \theta < 0, & \eta > 0 & \text{and} & \theta > 0, \quad \eta < 0. \end{array}$$

Any harmonic function is necessarily analytic in ξ, η and, hence, we may write $\theta = \eta\bar{\theta}$ where $\bar{\theta} = \bar{\theta}(\xi, \eta)$ is an analytic function in ξ, η on D which, by (3.2), does not change sign on $D + \partial D$. A straightforward calculation shows that $\bar{\theta}$ satisfies equation (3.1) in D and that for $\rho = 1$, by virtue of (1.1), $\bar{\theta}$ satisfies

$$\frac{\partial \bar{\theta}}{\partial \rho} = h(\sigma)\bar{\theta}, \qquad 0 \leqq \sigma \leqq 2\pi,$$

where σ is the polar angle of ξ, η and

$$h(\sigma) = \left[\frac{p_m e^{-3\tau} \sin \theta}{\theta} - 1\right]_{\rho=1}.$$

The function $h(\sigma)$ is continuously differentiable for all $0 \leqq \sigma \leqq 2\pi$. Lemma 3.2 (with $r = e^{\tau}$) together with (1.3) and $p_m \geqq (1 + \beta)^3$ implies $h(\sigma) \geqq 0$ and consequently, since $\bar{\theta}$ does not change sign on D, neither does $\partial\bar{\theta}/\partial\rho$. This contradicts Lemma 3.1 unless $\bar{\theta} \equiv k = $ const. in which case $\theta = k\eta$, $\tau = -k\xi$. But then (1.1) implies

$$k\sigma + \cdots = k \sin \sigma = \partial\theta/\partial\rho = p_m e^{-3\tau} \sin \theta = kp_m\sigma + \cdots$$

and, hence, $k = 0$ (since $p_m \geqq (1 + \beta)^3 > 1$). This leads us to the contradiction that $\omega \equiv 0$ and, consequently, the assumption $p_m \geqq (1 + \beta)^3$ is false.

References

[1] Courant & Hilbert, *Methods of Mathematical Physics*, Vol. II Interscience Publishers, Inc., New York, 1944, p. 252.

[2] J. Cushing, *Local uniqueness for harmonic functions under nonlinear boundary conditions*, U. of Maryland Doctoral Thesis, 1968.

[3] D. Dunninger & M. H. Martin, On a uniqueness question of Levi-Civita, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **41** (1966) 452-459.

[4] O. D. Kellogg, *Potential Theory*, Dover Publications, Inc., New York, 1953, p. 100.

[5] T. Levi-Civita, Determination rigourese des ondes permanentes d'ampleur finie, *Math. Ann.*, **93** (1925) 264-314.

[6] L. Lichtenstein, *Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integro-Differentialgleichungen nebst Anwendungen*, J. Springer, Berlin 1931.

[7] M. H. Martin, *Local Uniqueness in Boundary Problems*, Tech. Note BN-581, Inst. for Fluid Dynamics and Appl. Math., U. of Md., College Pk., Md., 1968.

[8] A. I. Nekrassov, *The Exact Theory of Steady State Waves on the Surface of a Heavy Liquid*, MRC Tech. Summary Report 813, U. of Wisc., Madison, Wisc., July, 1967.

[9] J. J. Stoker, *Water Waves*, Interscience Publishers, Inc., New York, 1957, pp. 522-543.

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