# *Some Existence Theorems for Nonlinear Eigenvalue Problems Associated with Elliptic Equations*

J. M. CUSHING

*Communicated by* J. SERRIN

## 1. Introduction

The existence of harmonic functions in a region $D$ of the plane satisfying nonlinear Neumann boundary conditions of the form $\partial u/\partial n = f(u)$ where $n$ is the outwardly directed unit normal to the boundary $\partial D$ of $D$ has been studied by T. CARLEMAN ([4]), NAKAMORI & SUYAMA ([16]), K. KLINGELHÖFER ([9], [10], [11]), and, in the case when $D$ is the unit disk and $f$ is of a highly specialized form which also depends on the harmonic conjugate of $u$, by LEVI-CIVITA ([13]) in his study of periodic progressing water waves. CARLEMAN and NAKAMORI & SUYAMA considered the case when $f'(u) < 0$ and found existence results for singular solutions. KLINGELHÖFER, using the contraction mapping principle, developed existence and uniqueness theorems for regular solutions under assumptions on $f(u)$ which, roughly speaking, require that $f'(u)$ stay away from the (necessarily positive) eigenvalues of the linear problem (the Steklov problem [2], [19]) $\partial u/\partial n = \lambda u$; that is, $\lambda_k < f'(u) < \lambda_{k+1}$ where $\lambda_k$, $\lambda_{k+1}$ are consecutive Steklov eigenvalues. These results, however, never yield nontrivial solutions to the problem in the case that $f(0) = 0$; LEVI-CIVITA, on the other hand, found (in his special problem) nontrivial solutions even though the problem had the trivial solution $u = 0$. The problem of finding nontrivial solutions to elliptic equations under such nonlinear boundary conditions (actually a problem in bifurcation theory) is the subject of this paper. Various criteria for uniqueness have been found by many authors including MARTIN, LEVIN, DUNNINGER, and CUSHING (see [5] for bibliography).

The problem is the following: to satisfy the equation

$$Lu = 0, \qquad x \in D \tag{1.1}$$

together with the boundary conditions

$$\frac{\partial u}{\partial \nu} = \lambda u + g(u, x), \qquad x \in \partial D \tag{1.2}$$

or

$$\frac{\partial u}{\partial \nu} = \lambda [u + g(u, x)], \qquad x \in \partial D \tag{1.2'}$$

where $D$ is a bounded region in Euclidean $m$ space, $E^m$, with boundary $\partial D$ of type $C^{1+\lambda}$, $0<\lambda<1$; $x=(x_1, x_2, \ldots, x_m)\in E^m$; here

$$Lu \equiv \sum_{i,j=1}^{m} D_i\big(a_{ij}(x) D_j u\big), \qquad D_i=\partial/\partial x_i$$

and

$$\frac{\partial u}{\partial \nu(x)} \equiv \sum_{i,j=1}^{m} a_{ij}(x)\, n_i(x)\, D_j u$$

where $n(x)=(n_1(x), n_2(x), \ldots, n_m(x))$ is the outwardly directed unit normal to $\partial D$ at $x\in\partial D$. Here $a_{ij}(x)\in C^{1+\lambda}(E^m)$, $0<\lambda<1$, satisfy $a_{ij}(x)=a_{ji}(x)$, $x\in E^m$, and

$$\sum_{i,j=1}^{m} a_{ij}(x)\,\xi_i\,\xi_j>0, \qquad \xi_1^2+\cdots+\xi_m^2 \neq 0, \qquad x\in E^m.$$

The constant $\lambda$ is to be determined as part of the solution while $g(u, x)$ is a given function such that $g(0, x)=0$, $x\in\partial D$; more conditions will be placed on $g$, $D$, and $a_{ij}(x)$ below. By a solution to (1.1) we will always mean a function $u(x)\in C^2(D)$, $u(x)\in C^0(\bar{D})$, $(\bar{D}=D+\partial D)$. Throughout this paper any equation involving the conormal derivative $\partial u/\partial \nu$ on $\partial D$ is meant in the sense that

$$\frac{\partial u(x_0)}{\partial \nu(x_0)} \equiv \lim_{\substack{x\to x_0\\ x\in C}} \frac{\partial u(x)}{\partial \nu(x_0)}, \qquad x_0\in\partial D$$

where $C$ is any finite closed cone with vertex $x_0$ which is contained in $D+\{x_0\}$.

The purpose of this paper is to prove, under suitable conditions on $L$ and on $g(u, x)$, the existence of nontrivial solutions to (1.1)–(1.2) and (1.1)–(1.2)′ for certain values of $\lambda$. It will be necessary to make certain symmetry assumptions on $a_{ij}$, $D$, and $g$, and the solutions with which we deal will be of a symmetric type. Keeping within the spirit of the general theory of bifurcation problems (especially the expansion techniques of LIAPUNOV and SCHMIDT [20]) we will look for solutions of small norm corresponding to values of $\lambda$ near the eigenvalues of the linearized problem $g(u)\equiv 0$ (the generalized Steklov problem). In §2 we briefly develop the necessary facts concerning the classical Neumann and Steklov problems for (1.1), and in §3 and §5 we give our main results (Theorems 3.1, 5.1, 5.2) for the nonlinear problems (1.1)–(1.2) and (1 1)–(1.2)′. The proof of Theorem 3.1 is completed in §4, while in §6 a simple example for Laplace's equation in the plane is given.

## 2. Preliminaries

In this section we develop the results we require concerning the linear problem associated with (1.1). Under the assumptions made in §1 there exists a fundamental solution of $Lu=0$ in any region $\Omega$ in $E^m$ of class $C^{1+\lambda}$ (cf. [7], p. 153; [15]); that is, there exists a function $\Gamma(x, y)$ defined for all $x, y\in\bar{\Omega}$, $x\neq y$, which as a function of $x$ satisfies for all $y\in\bar{\Omega}$ the equation $Lu=0$, $x\neq y$, and which, for some real $\mu>0$, satisfies

$$D_x^i[\Gamma(x, y)-\Gamma_0(x, y)]=O(|x-y|^{2-i-m+\mu}) \qquad (i=0, 1, 2)$$

where

$$\Gamma_0(x,y)=\begin{cases}\dfrac{[\Sigma a^{ij}(y)(x_i-y_i)(x_j-y_j)]^{(2-m)/2}}{(m-2)\,\omega_n[\det(a^{ij}(y))]^{1/2}} & \text{if } m>2\\[2ex] \dfrac{\log[\Sigma a^{ij}(y)(x_i-y_i)(x_j-y_j)]^{1/2}}{2\pi[\det(a^{ij}(y))]^{1/2}} & \text{if } m=2.\end{cases}$$

Here $(a^{ij}(y))$ is the inverse of the matrix $(a_{ij}(y))$ and $\omega_m=2\pi^{\frac{1}{2}m}/\Gamma(\frac{1}{2}m)$ is the surface area of the unit hypersphere in $E^m$. This fact together with the unique continuation principle for elliptic equations ([1]) implies the existence of $\Gamma(x,y)$ for all $x, y\in E^m$, $x\neq y$. Moreover, since (1.1) is self-adjoint, $\Gamma(x,y)=\Gamma(y,x)$ for $x\neq y$ ([15], p. 58). Using $\Gamma(x,y)$ it is possible to solve the Neumann problem for (1.1) and to construct a Neumann function by the Fredholm method of integral equations; the details, which we briefly sketch, closely parallel those for harmonic functions (see, for example, [8], [18]).

Let $\phi(x)\in C^0(\partial D)$ satisfy the condition

$$\int_{\partial D}\phi(y)\,dS_y=0 \tag{2.1}$$

where $dS_y$ = surface element on $\partial D$. The Neumann problem consists of finding a solution to (1.1) satisfying

$$\frac{\partial u}{\partial \nu}=\phi \quad \text{on } \partial D. \tag{2.2}$$

Condition (2.1) is necessary as is seen from the generalized Gauss theorem for solutions of (1.1)

$$\int_{\partial D}\frac{\partial u}{\partial \nu}\,dS_y=0,$$

which results from an easy application of the divergence theorem. A solution is sought in the form of a (generalized) single layer potential of density $\alpha(y)$:

$$u(x)=\int_{\partial D}\Gamma(x,y)\,\alpha(y)\,dS_y. \tag{2.3}$$

This solution to (1.1) satisfies the "jump" relation ([15])

$$\frac{\partial u(x)}{\partial \nu(x)}=\frac{1}{2}\alpha(x)-\int_{\partial D}\frac{\partial \Gamma(x,y)}{\partial \nu(x)}\,\alpha(y)\,dS_y$$

which together with (2.2) leads to the integral equation

$$\alpha(x)-2\int_{\partial D}\frac{\partial \Gamma(x,y)}{\partial \nu(x)}\,\alpha(y)\,dS_y=2\phi(x) \tag{2.4}$$

for $\alpha(x)$. The Fredholm alternate being valid for this equation, we must consider the homogeneous equation

$$\alpha(x)-2\int_{\partial D}\frac{\partial \Gamma(x,y)}{\partial \nu(x)}\,\alpha(y)\,dS_y=0, \tag{2.5}$$

a nontrivial solution to which yields a solution to the homogeneous, exterior Dirichlet problem in the form of a double layer potential

$$v(x)=\int_{\partial D}\frac{\partial\Gamma(x,y)}{\partial\nu(x)}\alpha(y)\,dS_y.$$

It follows ([18]) that $v(x)\equiv 0$ exterior to $D$; hence $\partial v/\partial\nu=0$ on $\partial D$ since the conormal of $v(x)$ is continuous across $\partial D$. But then $v\equiv$const. in $D$, and as $\alpha(x)$ equals the "jump" of $v(x)$ across $\partial D$ ([15], p. 34) it follows that $\alpha(x)\equiv$const. is the only nontrivial solution to (2.5). The Fredholm alternate now implies the existence of a solution $\alpha(x)$ to (2.4) for any $\phi(x)$ satisfying the orthogonality condition (2.1) and, thus, a solution to the Neumann problem as defined by (2.3).

A Neumann function may be constructed in the form

$$N(x,y)=\Gamma(x,y)+\psi(x,y) \tag{2.6}$$

where, for each fixed $x\in D$, $\psi(x,y)$ is a solution of (1.1) as a function of $y$ satisfying $\psi(x,y)\in C^2(D)$, $\psi(x,y)\in C^0(\bar{D})$, and

$$\frac{\partial\psi(x,y)}{\partial\nu(y)}=\beta(x,y)\equiv-\frac{\partial\Gamma(x,y)}{\partial\nu(y)}+C,\qquad y\in\partial D \tag{2.7}$$

where

$$C=S^{-1}\int_{\partial D}\frac{\partial\Gamma(x,y)}{\partial\nu(y)}\,dS_y,\qquad S=\text{surface area of }\partial D. \tag{2.8}$$

(It follows from Stokes' integral identity ([15], p. 15) that $C=$const.) Since $x\in D$, we have $\beta(x,y)\in C^0$ for $y\in\partial D$ and, moreover, $\int_{\partial D}\beta\,dS_y=0$ by (2.8); the existence of $\psi(x,y)$ is then assured. Now suppose $\phi(x)\in C^0(\partial D)$ satisfies (2.1). Then

$$u(x)=\int_{\partial D}N(x,y)\,\phi(y)\,dS_y \tag{2.9}$$

is the solution to the Neumann problem $\partial u/\partial\nu=\phi$ on $\partial D$ satisfying $\int_{\partial D}u(y)\,dS_y=0$, for if $u(x)$ is the solution to this problem we have from Stokes' identity and (2.7)

$$u(x)=\int_{\partial D}N(x,y)\frac{\partial u(y)}{\partial\nu(y)}\,dS_y-\int_{\partial D}\frac{\partial N(x,y)}{\partial\nu(y)}u(y)\,dS_y=\int_{\partial D}N(x,y)\,\phi(y)\,dS_y.$$

Since $\psi$ was determined only up to an additive function of $x$ we may assume

$$\int_{\partial D}N(x,y)\,dS_y=0. \tag{2.10}$$

Using standard arguments it is easily shown that $N(x,y)$ is symmetric in $x$ and $y$; that is, $N(x,y)=N(y,x)$ for $x\neq y$.

It can be shown ([15]) that $N(x,y)$ satisfies the condition

$$\lim_{x\to x_0}\int_{\partial D}|N(x,y)-N(x_0,y)|\,dS_y=0$$

for $x, x_0\in\bar{D}$, and hence the linear operator $A\phi\equiv\int_{\partial D}N(x,y)\,\phi(y)\,dS_y$, $x\in\partial D$, maps any Banach Space $B(\partial D)$ of continuous functions satisfying (2.1) under the norm

$\|\phi\|_{\partial D}=\max_{\partial D}|\phi|$ into the Banach space $C^0(\partial D)$ of continuous functions; moreover, this property implies that $A$ is a compact operator ([12], p. 19). We summarize these results in the lemma below.

**Lemma 2.1.** *The Neumann problem* (2.2) *for equation* (1.1) *has, up to an additive constant, a unique solution* (2.9) *for every* $\phi(x)\in C^0(\partial D)$ *satisfying* (2.1) *where* $N(x,y)$ *is the Neumann function for* $D$. *The operator*

(2.11) $$A\phi\equiv\int_{\partial D} N(x,y)\,\phi(y)\,dS_y, \qquad x\in\partial D$$

*is a compact, linear operator from* $B$ *into* $C^0$ *under the norm* $\|\phi\|=\max|\phi|$.

The linear problem associated with (1.2), (1.2)′ is given by the boundary condition

(2.12) $$\frac{\partial u}{\partial v}=\lambda u \quad \text{on } \partial D$$

where $\lambda$ is a real constant to be determined. In order to consider this problem and the nonlinear problems (1.2), (1.2)′, we will make certain assumptions on the domain $D$ and the coefficients $a_{ij}(x)$ in order to guarantee that the operator $A$ maps some $B(\partial D)$ into itself. We assume that $D$ has, for a suitable choice of coordinate axes, a reflective symmetry property and that the coefficients $a_{ij}(x)$ are either even or odd with respect to this reflection. More precisely, let $k$ be an integer $1\leqq k\leqq m$ and for $x=(x_1,x_2,\ldots,x_m)\in E^m$ let $-x$ denote the point $(-x_1,-x_2,\ldots,-x_k,x_{k+1},\ldots,x_m)\in E^m$. We assume also that for some fixed $k\geqq 1$:

(H1) $x\in D\Rightarrow -x\in D$;

(H2) $a_{ij}(-x)=a_{ij}(x), \quad x\in D,$

for $(i,j)\in\{[1,k]\times[1,k]\}\cup\{[k+1,m]\times[k+1,m]\}$; and

$$a_{ij}(-x)=-a_{ij}(x), \quad x\in D,$$

for $(i,j)\in\{[1,k]\times[k+1,m]\}\cup\{[k+1,m]\times[1,k]\}$.

**Lemma 2.2.** *Under hypotheses* H1, H2, *the compact operator* $A$ *maps the Banach Space* $O(\partial D)=\{\psi\,|\,\psi\in C^0(\partial D),\,\psi(-x)=-\psi(x)\}$ *into itself.*

Clearly H1 and the definition of $O(\partial D)$ imply that $\psi\in O(\partial D)$ is a solution of (2.1); hence $A\psi$ is the boundary value of the solution $u(x)$ to (1.1)–(2.2) which satisfies $\int_{\partial D} u(x)\,dS_x=0$. Also H1, H2 imply that $v(x)\equiv u(-x)$ satisfies (1.1), the boundary condition $\partial v/\partial v=-\psi(x)$ (here we have used $\psi(-x)=-\psi(x)$), and the condition $\int_{\partial D} v(x)\,dS_x=0$. But since $-u(x)$ also satisfies the latter problem, it follows that $v(x)=-u(x)+c$, $c=$const.; since $\int_{\partial D} v(x)\,dS_x=\int_{\partial D} u(x)\,dS_x=0$ the constant $c$ must be zero. This means $u(-x)=-u(x)$, $x\in\bar{D}$, and thus the boundary values $A\psi$ of $u(x)$ are in $O(\partial D)$. This completes the proof of the lemma.

Returning to the Steklov problem (2.12), it is easy to see that a nontrivial solution in $O(\partial D)$ exists only for a countable set of positive, real values of $\lambda$ of finite multiplicity, called the Steklov eigenvalues of $L$ on $D$. We need only notice that problem (1.1)–(2.12) in $O(\partial D)$ is, according to (2.11), equivalent to solving the following integral equation for the boundary values of $u(x) \in O(\partial D)$ ($u(x)$ being uniquely determined by its boundary values):

$$u(x)=\lambda \int_{\partial D} N(x, y) u(y) dS_y, \quad x \in \partial D.$$

In operator notation this is the problem of finding the eigenvalues and eigenfunctions of the linear compact operator $A$ on $O(\partial D)$ ([6], p. 579). Moreover, these eigenvalues are real and positive as can be seen by the integral identity (valid for solutions of (1.1)–(2.12))

$$\lambda \int_{\partial D} u^2 dS_y = \int_{\partial D} u \frac{\partial u}{\partial \nu} dS_y = \int_D \sum a_{ij}(y) D_i u D_j u \, dy$$

and the positive definiteness of the matrix $(a_{ij}(y))$.

We turn now to the nonhomogeneous Steklov problem

$$\frac{\partial u}{\partial \nu}=\lambda u+\psi \quad \text{on } \partial D \tag{2.13}$$

where $\psi \in O(\partial D)$. Using (2.11) we see that solving the problem (1.1)–(2.13) in $O(\partial D)$ is equivalent to the nonhomogeneous integral equation

$$u(x)=\lambda \int_{\partial D} N(x, y) u(y) dS_y + \Psi(x), \quad x \in \partial D \tag{2.14}$$

for the boundary values of $u(x)$ in $O(\partial D)$ where

$$\Psi(x)=\int_{\partial D} N(x, y) \psi(y) dS_y, \quad x \in \partial D \tag{2.15}$$

is continuous on $\partial D$. The Fredholm alternative in $O(\partial D)$ is valid for this equation, and, consequently, there exists a unique solution to (1.1)–(2.13) if $\lambda$ is not a Steklov eigenvalue. If $\lambda$ is a Steklov eigenvalue, however, then (1.1)–(2.13) has a solution if and only if

$$\int_{\partial D} \Psi(y) u_i(y, \lambda) dS_y = 0, \quad (i=1, 2, \dots, m(\lambda)) \tag{2.16}$$

where $u_i(y, \lambda)$ are the eigenfunctions in $O(\partial D)$ associated with $\lambda$ and $m(\lambda)$ is the multiplicity of $\lambda$. This orthogonality condition can be rewritten as a condition on $\psi$ by substituting (2.15) into (2.16), using the symmetry of $N(x, y)$ and applying Fubini's theorem. This leads to

$$\int_{\partial D} \psi(x) \Big[\int_{\partial D} N(y, x) u_i(y, \lambda) dS_y\Big] dS_x = 0.$$

But $\int N(y, x) u_i(y, \lambda) dS_y = \lambda^{-1} u_i(x, \lambda)$ so that the orthogonality condition becomes

$$\int_{\partial D} \psi(y) u_i(y, \lambda) dS_y = 0, \quad (i=1, 2, \dots, m(\lambda)). \tag{2.17}$$

We summarize these results in the following lemma.

**Lemma 2.3.** *The Steklov problem* (1.1)–(2.12) *on* $O(\partial D)$ *has a countable number of positive eigenvalues of finite multiplicity. For each* $\psi \in O(\partial D)$ *the nonhomogeneous problem* (1.1)–(2.13) *has a unique solution with boundary values in* $O(\partial D)$ *provided* $\lambda$ *is not a Steklov eigenvalue of* $L$; *if* $\lambda$ *is a Steklov eigenvalue, the* (1.1)–(2.13) *has a solution in* $O(\partial D)$ *if and only if the orthogonality conditions* (2.17) *hold for all eigensolutions* $u_i(y, \lambda) \in O(\partial D)$ *corresponding to* $\lambda$.

Before turning to the nonlinear problem (1.1)–(1.2) we state and prove an a priori estimate for solutions to the nonhomogeneous linear problem (1.1)–(2.13) which we will require in §4. If $f(x)$ is a function continuous on a closed region $S$, we denote $\|f\|_S = \max_{x \in S} |f(x)|$.

**Lemma 2.4.** *Let* $\lambda$ *be a Steklov eigenvalue for* (1.1) *of multiplicity* $m(\lambda)$ *and let* $u_i(x, \lambda) \in O(\partial D)$, $i = 1, 2, \ldots, m(\lambda)$, *be the corresponding eigenfunctions (which are normalized in some manner). Let* $\psi \in O(\partial D)$ *satisfy* (2.17) *and let* $u(x)$ *be the solution of* (1.1)–(2.13) *in* $O(\partial D)$ *satisfying the conditions*

$$\int_{\partial D} u(y)\, u_i(y, \lambda)\, dS_y = l_i \qquad (i = 1, 2, \ldots, m(\lambda)) \tag{2.18}$$

*where* $l = (l_1, l_2, \ldots, l_{m(\lambda)})$ *is an arbitrary, but fixed point in* $R^{m(\lambda)}$. *Then there exists a constant* $C > 0$ *such that*

$$\|u\|_{\bar{D}} \leqq C(|l| + \|\psi\|_{\partial D}) \tag{2.19}$$

*where* $|l| = \sum_{i=1}^{m(\lambda)} |l_i|$.

**Proof.** Define the linear operator $T$: $D(T) = O(\partial D) \times R^{m(\lambda)} \to C^0(\bar{D})$ by $T([\psi, l]) = u$ for $[\psi, l] \in O(\partial D) \times R^{m(\lambda)}$, where $u$ is the unique solution of (1.1)–(2.13) satisfying the conditions (2.18). We now show that $T$ is a closed operator on $D(T)$ under the norm $\|[\psi, l]\| = \|\psi\|_{\partial D} + |l|$ where $|l| = \sum_{i=1}^{m(\lambda)} |l_i|$; $D(T)$ is clearly a Banach space with respect to this norm. Suppose $[\psi_n, l^n] \in D(T)$ for $n = 1, 2, \ldots,$ and $\|[\psi_n, l^n] - [\psi, l]\| \to 0$ and $\|u_n - u\|_{\bar{D}} \to 0$ as $n \to \infty$ for some $[\psi, l] \in D(T)$ and $u \in C^0(\bar{D})$. From the inequalities

$$\left| \int_{\partial D} N(x, y)\, u_n(y)\, dS_y - \int_{\partial D} N(x, y)\, u(y)\, dS_y \right| \leqq \int_{\partial D} |N(x, y)|\, dS_y \cdot \|u_n - u\|_{\partial D},$$

$$|\Psi_n(x) - \Psi(x)| \leqq \int_{\partial D} |N(x, y)|\, dS_y \cdot \|\psi_n - \psi\|_{\partial D}$$

where $u_n = T([\psi_n, l^n])$, we see from (2.14) and (2.15) that

$$u_n(x) = \lambda \int_{\partial D} N(x, y)\, u_n(y)\, dS_y + \Psi_n(x), \qquad x \in \partial D$$

implies (in the limit as $n \to \infty$) the equation

$$u(x) = \lambda \int_{\partial D} N(x, y)\, u(y)\, dS_y + \Psi(x), \qquad x \in \partial D;$$

that is, $u$ solves (1.1)–(2.13) with data $\psi$. Also (2.18) for $u_n$ and $l^n$ states that

$$\int_{\partial D} u_n(y)\, u_i(y,\lambda)\, dS_y = l_i^n \qquad (i=1,2,\dots,m(\lambda))$$

and this implies as $n\to\infty$ (and, hence, $l_i^n \to l_i$ and $u_n \to u$ uniformly on $\partial D$) that

$$\int_{\partial D} u(y)\, u_i(y,\lambda)\, dS_y = l_i \qquad (i=1,2,\dots,m(\lambda)).$$

Thus, $([\psi, l], u)$ lies on the graph of $T$; that is, $T([\psi, l]) = u$ and $T$ is closed. The closed graph theorem ([6]) implies $T$ is continuous and consequently (2.19) holds for some constant $C>0$.

## 3. The Nonlinear Problem

In this section and the next we will state and prove our main result which is contained in the theorem below. We will need the following assumptions:

(H3) $g(u,x)$ is continuous on $(-\infty,+\infty)\times\partial D$ and analytic in $u$ at $u=0$ for each $x\in\partial D$;

(H4) $g(-u,x)=-g(u,x)$, $g(u,-x)=g(u,x)$ on $(-\infty,+\infty)\times\partial D$ (where $D$ is assumed to satisfy (H1) for some $k$);

(H5) $g(0,x)=0$, $\partial g(0,x)/\partial u=0$ for all $x\in\partial D$.

**Theorem 3.1.** *Assume hypotheses* (H1)–(H5) *and also that $\lambda_1$ is a Steklov eigenvalue of $L$ on $D$ of multiplicity one in the space $O(\partial D)$. Then for each real number $\alpha$ and each real $\varepsilon$ for which $|\varepsilon|$ is sufficiently small, there exists a unique non-trivial solution to* (1.1)–(1.2), (1.1)–(1.2)′ *of the form*

$$u(x)=\sum_{k=1}^{\infty} u_k(x)\,\varepsilon^k \tag{3.1}$$

*with boundary values in $O(\partial D)$ for*

$$\lambda=\sum_{k=1}^{\infty} \lambda_k\,\varepsilon^{k-1} \tag{3.2}$$

*where $u_1(x)$ is an eigenfunction associated with $\lambda_1$ (which is normalized in some manner), the $u_k$, $k\geqq 2$, are solutions to* (1.1) *with boundary values in $O(\partial D)$ satisfying*

$$\int_{\partial D} u_1(y)\, u_k(y)\, dS_y = \alpha, \qquad k\geqq 2, \tag{3.3}$$

*and the $\lambda_k$ are uniquely defined constants. The convergence of* (3.1) *is uniform in $\bar{D}$.*

To prove this theorem we formally substitute the series (3.1) and (3.2) into the nonlinear boundary condition (1.2), (1.2)′ and equate like powers of $\varepsilon$; this will lead to a nonhomogeneous Steklov problem for $u_k(x)$. The constants $\lambda_k$ will then be chosen so that the orthogonality condition (2.17) holds and as a result the existence and uniqueness of $u_k$ will be guaranteed by Lemma 2.2 and condition (3.3). The proof of the convergence of the series is based on the a priori estimate of Lemma 2.3 and will be carried out in §4 for the problem (1.2). The proof for problem (1.2)′ is very similar and will not be given.

Under assumptions (H3)–(H5) we may write $g(u,x)=\sum_{k=2}^{\infty} g_k(x)\,u^k$ where $g_{2k}(x)=0$, $k\geqq 1$, and $g_{2k+1}(-x)=g_{2k+1}(x)$, $k\geqq 0$, $x\in\partial D$. Substituting the series (3.1) for $u(x)$ into the expansion for $g$ we obtain

$$g(u,x)=g\left(\sum_{k=1}^{\infty} u_k\,\varepsilon^k, x\right)=\sum_{k=3}^{\infty} G_k\,\varepsilon^k$$

where

$$G_k=G_k(u_i;\,g_{i+1}),\qquad k\geqq 3, \tag{3.4}$$

is a polynomial of odd degree terms in the variables $u_i$ with coefficients $g_{i+1}$ $(i=1,2,\ldots,k-2)$. Thus, from (3.1) and (3.2), we find (setting $G_2=0$)

$$\lambda u+g(u,x)=\lambda_1 u_1+\sum_{k=2}^{\infty}\left[\lambda_1 u_k+\sum_{l=1}^{k-1}\lambda_{l+1}\,u_{k-l}+G_k\right]\varepsilon^k.$$

By equating like powers of $\varepsilon$ in (1.2) we obtain

$$\begin{aligned}\frac{\partial u_1}{\partial v}&=\lambda_1 u_1,\\ \frac{\partial u_k}{\partial v}&=\lambda_1 u_k+f_k,\qquad k\geqq 2,\end{aligned} \tag{3.5}$$

where

$$f_k=\lambda_k u_1+H_k,\qquad k\geqq 2, \tag{3.6}$$

$$H_2=0,\qquad H_k=\sum_{l=1}^{k-2}\lambda_{l+1}\,u_{k-l}+G_k,\qquad k\geqq 3. \tag{3.7}$$

Because of (H1) and (H4), if $u_1,\ldots,u_{k-1}$ have boundary values in $O(\partial D)$, then so does $H_k$ and consequently

$$\int_{\partial D} f_k\,dS_y=0. \tag{3.8}$$

From (3.5) we see that $u_1$ must be a Steklov eigenfunction corresponding to the eigenvalue $\lambda_1$, which we normalize in some manner. Since (3.8) holds, the boundary conditions (3.5) and conditions (3.3) recursively define the boundary values of the functions $u_k\in O(\partial D)$ uniquely as solutions to equation (1.1) provided the orthogonality condition (2.17) holds; that is,

$$\int_{\partial D} f_k\,u_1(y)\,dS_y=0,\qquad k\geqq 2.$$

From (3.6) we see that this condition can be fulfilled by choosing

$$\lambda_k=-K_1\int_{\partial D} H_k\,u_1\,dS_y,\qquad k\geqq 2 \tag{3.9}$$

where

$$K_1^{-1}=\int_{\partial D} u_1^2\,dS_y>0;$$

this uniquely defines the constants $\lambda_k$, $k\geqq 2$.

We now turn to the convergence of the series (3.2) and (3.3).

## 4. Convergence of the Series Expansions

Let $K_2 = \|u_1\|_{\bar{D}}$. Lemma 2.3 applied to the solutions $u_k$, $k \geqq 2$, satisfying (3.3) and (3.5) yields the estimate

$$\|u_k\|_{\partial D} \leqq C(|\alpha| + K_2 |\lambda_k| + \|H_k\|_{\partial D}).$$

From (3.9) we have

$$|\lambda_k| \leqq K_1 K_2 S \|H_k\|_{\partial D} \tag{4.1}$$

where $S$ is the total surface area of $\partial D$, and as a result

$$\|u_k\|_{\partial D} \leqq K_3 + K_4 \|H_k\|_{\partial D}, \quad k \geqq 2, \tag{4.2}$$

where $K_3$, $K_4$ are constants independent of $k$. Letting $p_k = \|u_k\|_{\partial D} + |\lambda_K|$ we may combine the estimates (4.1) and (4.2) to obtain

$$0 < p_k \leqq K_3 + K_5 \|H_k\|_{\partial D} \tag{4.3}$$

where $K_5$ is a constant independent of $k$. Now we obtain a bound for $H_k$. From (3.4) and the triangle inequality for norms we find

$$\|G_k\|_{\partial D} \leqq G(\|u_i\|_{\partial D}; \|g_{i+1}\|) \leqq G(p_i; q_{i+1}), \quad k \geqq 3,$$

where we have set $q_i = \|g_i\|$, $i \geqq 2$, and consequently

$$\|H_k\|_{\partial D} \leqq \sum_{l=1}^{k-2} p_{l+1}\, p_{k-l} + G_k(p_i; q_{i+1}), \quad k \geqq 3. \tag{4.4}$$

Let $g^*(\varepsilon) = \sum_{k=2}^{\infty} q_k \varepsilon^k$ and consider the function of two real variables $z$, defined by

$$h(z, \varepsilon) = p_2 \varepsilon + K_3 \sum_{k=2}^{\infty} \varepsilon^k - z + K_5 z^2 + K_5 \varepsilon^{-1} g^*(z\varepsilon)$$

for $z, \varepsilon$ sufficiently small. Since $h(0, 0) = 0$, $h_z(0, 0) = -1$, the implicit function theorem asserts that the equation $h(z, \varepsilon) = 0$ has, in a sufficiently small neighborhood of the point $z = 0$, $\varepsilon = 0$, a unique, analytic solution

$$z = \sum_{k=0}^{\infty} z_{k+1} \varepsilon^k \tag{4.5}$$

passing through the point $z = 0$, $\varepsilon = 0$. Thus $z_1 = 0$, and then by implicit differentiation we find $z_2 = p_2$. The equation $h(z, \varepsilon) = 0$ is equivalent to

$$z - p_2 \varepsilon = K_3 \sum_{k=2}^{\infty} \varepsilon^k + K_5 z^2 + K_5 \varepsilon^{-1} g^*(z\varepsilon)$$

which, upon substitution of (4.5), yields

$$\sum_{k=2}^{\infty} z_{k+1} \varepsilon^k = \sum_{k=2}^{\infty} \left\{ K_3 + K_5 \left( \sum_{l=1}^{k-1} z_{l+1} z_{k-l+1} + G_{k+1} \right) \right\} \varepsilon^k$$

and, finally, the recursive formula for $k \geqq 2$,

$$z_{k+1} = K_3 + K_5 \left\{ \sum_{l=1}^{k-1} z_{l+1} z_{k-l+1} + G_{k+1}(z_i; q_{i+1}) \right\} \tag{4.6}$$

for the coefficients of (4.5). Since $z_2 = p_2 > 0$, an easy induction shows (using (4.3) and (4.4))

$$z_{k+1} \geqq K_3 + K_5 \left\{ \sum_{l=1}^{k-1} p_{l+1}\, p_{k-l+1} + G_{k+1} \right\}$$

$$\geqq K_3 + K_5 \, \|H_{k+1}\|_{\partial D} \geqq p_{k+1} > 0, \qquad k \geqq 2.$$

By the comparison test for series it follows that $\sum p_k{}^k$ converges, and hence (since $p_k = \|u_k\|_{\partial D} + |\lambda_k|$) the series (3.1) and (3.2) also converge (uniformly in $\bar{D}$ in the case of (3.1)) for $|\varepsilon|$ sufficiently small. The proof of Theorem 3.2 is now complete.

## 5. Further Theorems for (1.1)–(1.2)′

Theorem 3.1 gives a constructive existence proof for problems (1.1)–(1.2), (1.2)′ but requires the analyticity of $g$ in $u$. In this section we offer two theorems for (1.1)–(1.2)′ which considerably weaken this differentiability condition on $g$. The first (Theorem 5.1), however, is nonconstructive in that it depends on the purely topological techniques of Krasnosel'skii ([12]). Theorem 5.2 uses the standard bifurcation methods of Schmidt (ref. [17], [20]) in the case of a simple Steklov eigenvalue $\lambda_1$ and is given here because of its information concerning the nature of the bifurcation at $\lambda_1$.

We use the term *bifurcation point* in the sense of Krasnosel'skii ([12], p. 181); that is, $\lambda_1$ is a bifurcation point of an operator $B$ which maps a Banach space $E$ into itself ($B0=0$) if for any $\varepsilon$, $\delta > 0$ there exists an eigenvalue $\lambda$ of $B$ such that $|\lambda - \lambda_1| < \varepsilon$ and an eigenfunction $\psi \in E$: $\psi = \lambda B \psi$ with norm less than $\delta$. The eigenfunctions $\psi$ of $B$ form a *continuous branch* in the neighborhood of $(\lambda_1, 0)$ if the boundary of each open ball in $E$ centered at 0 has a non-empty intersection with the set of eigenfunctions.

We are now ready to state our most general theorem for (1.1)–(1.2)′. In place of (H3) we need

(H3′) $g(u, x)$ is continuous on $(-\infty, +\infty) \times \partial D$ and continuously differentiable with respect to $u$ for all $x \in \partial D$.

**Theorem 5.1.** *If* (H1), (H2), (H3′), (H4), (H5) *are all satisfied, then each Steklov eigenvalue* $\lambda_1$ *of odd multiplicity in* $O(\partial D)$ *is a bifurcation point in* $O(\partial D)$ *for problem* (1.1)–(1.2)′; *moreover, to each such eigenvalue there corresponds a continuous branch of eigenfunctions.*

This theorem is a direct consequence of Theorem 2.1 in Krasnosel'skii ([12], p. 196) when applied to the nonlinear operator $B$ defined by $B\psi = Af\psi$, $\psi \in O(\partial D)$, where $f\psi = \psi + g(\psi, x)$ is, under the stated hypotheses, a continuous and bounded operator ([12], p. 32) of $O(\partial D)$ into itself. Since $A$ is compact it follows ([12], p. 46) that $B$ is continuous and compact from $O(\partial D)$ into itself. Clearly $B0=0$. Now $Bh = Ah + Rh$ where

$$Rh = \int_{\partial D} N(x, y)\, g(h, y)\, dS_y, \qquad h \in O(\partial D),$$

and $\|Rh\|_{\partial D}/\|h\|_{\partial D}\to 0$ as $\|h\|_{\partial D}\to 0$ since by (H5) $\|g(h,x)\|_{\partial D}/\|h\|_{\partial D}\to 0$. Consequently $A$ is the Frechét derivative of $B$ at $\psi=0$ and KRASNOSEL'SKII's theorem now applies. The theorem is thus proved.

In presenting our next theorem we follow the standard bifurcation techniques for compact operators on Banach spaces as presented by PIMBLEY ([17]). It is necessary to assume that $g(u, x)$ satisfies

(H3″) $g(u,x)$ is $k$ times continuously differentiable with respect to $u$ for each $x\in\partial D$ for some integer $k\geqq 3$, and satisfies $\partial g^l(0,x)/\partial u^l\equiv 0$, $x\in\partial D$, $2\leqq l<k$, $g^{(k)}(0,x)\not\equiv 0$ where $g^{(k)}(0,x)\equiv\partial^k g(0,x)/\partial u^k$.

Let $u_1$ be eigenfunction corresponding to a Steklov eigenvalue $\lambda_1$ of multiplicity one normalized by $\int_{\partial D} u_1^2\,dS_x=1$. It can be shown ([17]) that the solution of the eigenvalue problem $\lambda Bu=u$ for $\lambda$ sufficiently close to $\lambda_1$ and $u$ of sufficiently small norm is equivalent to solving the bifurcation equation

$$\delta[-1+\phi_1(\delta,\xi)]+\xi^2[C+\phi_2(\delta,\xi)]=0 \tag{5.1}$$

for $\xi$ as a function of $\delta$ in the neighborhood of $\xi=\delta=0$. Here $C$ is a constant given by

$$C=\frac{1}{k!}\lambda_1\int_{\partial D} g^{(k)}(0,x)\,u_1^{k+1}(x)\,dS_x \tag{5.2}$$

and $\phi_i\to 0$ as $\xi^2+\delta^2\to 0$, $i=1,2$. We have used here the compactness and symmetry of the operator $A$ on $O(\partial D)$ and the condition (H4) which implies $B$ is an odd operator; that is, $B(-u)=-Bu$, $u\in O(\partial D)$. Each solution of (5.1) $\xi=\xi(\delta)$ in the neighborhood of $\xi=\delta=0$ gives rise to a unique solution $u$ (by a contraction mapping) to (1.1)–(1.2)′ for $\lambda=\lambda_1+\delta$ (ref. [17]). It can be shown ([17], Chapter 4) by using Newton's Polygon method that there exist two branches of solutions $\xi=\xi(\delta)$ differing only in sign for $\delta>0$ if $C>0$ and $\delta<0$ if $C<0$. This leads to the following theorem (ref. [17], p. 34 and p. 126):

**Theorem 5.2.** *Assume* (H1), (H2), (H3″), (H4), (H5) *are all satisfied and that $\lambda_1$ is a Steklov eigenvalue of multiplicity one in $O(\partial D)$. Then in the neighborhood of $\lambda=\lambda_1$, $u=0$ there exist exactly two (continuous) branches of solutions to* (1.1)–(1.2)′ *differing only in sign. If $C$ as given in* (5.2) *is nonzero and positive (negative), then the branches exist only for $\lambda<\lambda_1$ $(>\lambda_1)$ and the bifurcation is said to be to the left (right).*

## 6. An Example

As a simple example we take $L=\Delta$ with $m=2$ (Laplace operator in the $x,y$ plane) and $D$ as the unit circle centered at the origin; the conormal derivative in (1.2), (1.2)′ then becomes the radial derivative of $u$. Hypotheses (H1), (H2) are satisfied with $k=1$ (that is, $D$ is symmetric with respect to the $y$-axis), and we may take

$$O(\partial D)=\{\psi(\theta)\,|\,\psi\in C^0[-\pi/2,3\pi/2],\quad \psi(\pi-\theta)=-\psi(\theta)\}$$

where $\theta$ is the polar angle measured from the $x$-axis. It is well-known ([3], [14]) that the Steklov problem for Laplace's equation on the unit circle has eigenvalues

$\lambda_n = n$ $(n=1, 2, \ldots)$ each of which has *two* linear independent solutions $r^n \sin n\theta$ and $r^n \cos n\theta$, where $(r, \theta)$ are polar coordinates in the plane. Within the space $O(\partial D)$, however, each eigenvalue is of multiplicity one, the corresponding eigensolution being $r^n \cos n\theta$ for $n$ odd or $r^n \sin n\theta$ for $n$ even; consequently all theorems of §3 and §5 are applicable provided the nonlinear term $g(u, \theta)$ meets the necessary requirements. For example, the problem $u + g(u, \theta) = \sin u$ in (1.1)–(1.2)′ may be taken as a nonlinear approximation to LEVI-CIVITA's ([13]) water wave theory. (Here a multiplicative factor of $e^{-3v}$ where $v$ is the harmonic conjugate of $u$ vanishing at the origin has been dropped from the boundary condition (1.2)′.) Since $g = \sin u - u$ satisfies the hypothesis (H3) (and hence (H3′) and (H3″) with $k=3$) as well as (H4) and (H5), the results of §3 and §5 imply that there exist exactly two continuous branches of eigensolutions differing only in sign bifurcating from $\lambda_n = n$, $u=0$ for each $n=1, 2, \ldots$. Moreover, since $g^{(3)}(0, \theta) = -1$, we find that the constant $C$ in (5.2) is negative at each eigenvalue; consequently, the bifurcation is to the right. (It should be pointed out however that the bifurcation in the LEVI-CIVITA problem is to the left; hence this approximate theory does not fully reflect the bifurcation behavior of that problem.)

## References


1. ARONSZAJIN, N., A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pures Appl. **36**, 235–249 (1957).
2. BERGMAN, S., & M. SCHIFFER, Kernel Functions and Elliptic Differential Equations in Mathematical Physics. New York: Academic Press Inc. 1953.
3. BOGGIO, T., Sulle funzioni di variable complessa in un'area circolari. Rend. della R. Academia de Torino **47**, 22–37 (1912).
4. CARLEMAN, T., Über eine nichtlineare Randwertaufgabe bei der Gleichung $\Delta u = 0$. Math. Z. **93**, 35–43 (1921).
5. CUSHING, J.M., A uniqueness criterion for harmonic functions under nonlinear boundary conditions. To appear.
6. DUNFORD, N., & J.T. SCHWARTZ, Linear Operators I: general theory. Pure and Appl. Math. Vol. 7. New York: Interscience 1958.
7. FRIEDMAN, A., Partial Differential Equations of Parabolic Type. New Jersey: Prentice-Hall Inc. 1964.
8. KELLOG, O.D., Foundations of Potential Theory. New York: Frederick Ungar Publishing Co. 1929.
9. KLINGELHÖFER, K., Über nichtlineare Randwertaufgaben der Potentialtheorie. Mitt. Math. Sem. Giessen **76**, 1–27 (1967).
10. KLINGELHÖFER, K., Nonlinear harmonic boundary value problems I. Arch. Rat. Mech. Anal. **31**, 364–371 (1968).
11. KLINGELHÖFER, K., Nonlinear harmonic boundary value problems II (modified Hammerstein integral equations). J. Math. Anal. Appl. **25**, 592–606 (1969).
12. KRASNOSEL'SKII, M.A., Topological Methods in the Theory of Nonlinear Integral Equations. International Series of Monographs on Pure and Appl. Math. Vol. 45. New York: Pergamon Press, MacMillan Co. 1964.
13. LEVI-CIVITA, T., Détermination rigourese des ondes permanentes d'ampleur fini. Math. Ann. **93**, 264–314 (1925).
14. MARTIN, M.H., Some aspects of uniqueness for solutions to boundary problems. Proc. Edinburgh Math. Soc. **13**, 25–35 (1962).
15. MIRANDA, C., Equazioni alle derivate parziali di tipo ellittico. Berlin-Göttingen-Heidelberg: Springer 1955.

16. NAKAMORI, K., & Y. SUYAMA, On a nonlinear boundary problem for the equations $\Delta u=0$ and $\Delta u=F(x, h)$. Mem. Fac. Sci. Kyusyu Univ. A 5, 99–106 (1950).
17. PIMBLEY, G.H., JR., Eigenfunction Branches of Nonlinear Operators, and Their Bifurcations. Lecture Notes in Mathemarics No. 104, Berlin-Heidelberg-New York: Springer 1969.
18. SOBOLEV, S.L., Partial Differential Equations of Mathematical Physics. New York: Pergamon Press 1964.
19. STEKLOV, M.W., Sur les problèmes fondamentaux de la physique mathématique. Ann. Sci. Ecole Norm. Sup. **19**, 455–490 (1902).
20. VAINBERG, M.M., & V.A. TRENOGIN, The methods of Liapunov and Schmidt in the theory of nonlinear equations and their further development. Russ. Math. Surveys **17**, 1–60 (1962).

Department of Mathematics
The University of Arizona
Tucson

*(Received January 26, 1971)*