JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 60, 747–754 (1977)

Stable Positive Periodic Solutions of the Time-Dependent Logistic Equation under Possible Hereditary Influences

J. M. CUSHING

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

Submitted by J. Cronin

The logistic equation, generalized to include time-dependent but periodic coefficients and a functional, hereditary interaction term, is shown to have a positive periodic solution provided the time-dependent net birth rate has a positive average. Under more restrictive conditions on the interaction term and the net birth rate, this solution is shown to be uniformly asymptotically stable. The approach is to treat the problem as one of the bifurcation of nontrivial positive solutions from the identically zero solution using, roughly speaking, the average of the net birth rate as a nonlinear eigenvalue.

1. INTRODUCTION

The well-known logistic equation,

$$N' = N(b - cN), \qquad ' = d/dt, \tag{1.1}$$

for positive constants b and c has a uniformly asymptotically stable equilibrium $N = b/c$. In fact, all solutions with positive initial values tend to b/c as $t \to +\infty$. These facts are, of course, easily seen from the general solution of (1.1) which can be obtained by elementary integration techniques. Equation (1.1) is most often thought of as a simple model for the growth of a single population whose size is measured in some appropriate units by $N(t)$; b is the net birth rate (i.e., the birth rate per unit N per unit time minus the death rate per unit N per unit time); c is an interaction coefficient which serves as a measure of the inhibiting effect upon the growth rate caused by population size; and b/c is often called the carrying capacity of the population (or of its environment).

As early as the work of Volterra [9], time lags or hereditary effects (continuously distributed time lags) have been considered in models of the form (1.1). More recent work is contained in [2, 4, 5, 7]. These models can be placed in the general form

$$N' = N\left(b - \int_0^\infty N(t - s)\, d\alpha(t, s)\right), \tag{1.2}$$

ISSN 0022-247X

where the integral expression measures the accumulated lag effects on the inhibiting term. If $\alpha(t, s) = cu_\tau(s)$, where u_τ is the unit step function at $\tau \geqslant 0$, then (1.2) reduces to an equation with a single time lag at τ (if $\tau = 0$, we get (1.1)); this equation, for b and c constant, has been studied in [4, 5]. If $\alpha(t, s) = c \int_0^s k(u)\, du$ for c constant, then (1.2) reduces to the hereditary equation studied in [2, 7, 9] (again b is a constant). If b and α are independent of t then (1.2) has an equilibrium $N = b / \int_0^\infty d\alpha(s)$; the above-mentioned references study the question of when this equilibrium is stable [7] or when in fact other (nonequilibrium) oscillations occur [2, 4, 5].

The purpose of this paper is to consider the general functional differential equation

$$N' = N(b(t) - H_t N), \tag{1.3}$$

where the net birth rate b is a periodic function of time (as might result from any number of biological or seasonal environmental causes) and NH_tN is a functional which describes the cumulative inhibiting effects due to the population sizes of possibly all past times. Sufficient conditions are given in Theorem 1 which ensure the existence of a positive periodic solution. To the author's knowledge, no such results are presently known for this general time-dependent, functional equation. The stability of solutions is considered in Theorems 2 and 3.

2. Results

Let B be the Banach space of all continuous, ω-periodic functions under the supremum norm: $|N|_0 = \max_{0 \leqslant t < \infty} |N(t)|$. Throughout this paper ω is an arbitrary but fixed period. The functional differential equation (1.3) is considered for $b \in B$, where $N \to H_t N$ is an operator which is defined and continuous in some neighborhood $\Omega \subseteq B$ of the origin $0 \in \Omega$ and which satisfies $H_t N = O(|N|_0)$ near $N = 0$. More is assumed about H_t below. By a solution N in Ω of (1.3) is meant a continuously differentiable function $N \in \Omega$ such that (1.3) is satisfied for all t. By a positive solution in Ω of (1.3) is meant a solution in Ω for which $N(t) > 0$ for all t. A positive function $N \in B$ is, of course, necessarily bounded away from zero: $N(t) \geqslant \alpha > 0$ for all t and some constant $\alpha > 0$. Lemma 1 in Section 3 shows that solutions in Ω of (1.3) which are positive at some time are positive for all time.

Let $[b]$ denote the average of $b(t)$:

$$[b] = \omega^{-1} \int_0^\omega b(s)\, ds.$$

Let $p(t) = b(t) - [b]$. Then $[p] = 0$ and $b \in B$ implies $p \in B$.

THEOREM 1. (a) *Let* $p \in B$, $[p] = 0$. *Assume the following:*

$$\text{The operator } N \to H_t N \text{ is continuous from } \Omega \text{ to } B,\ \Omega \text{ open in } B,\ 0 \in \Omega, \text{ and maps bounded sets to bounded sets with } H_t N = O(|N|_0) \text{ near } N = 0. \text{ In addition } [H_t N] > 0 \text{ for } N > 0,\ N \in \Omega. \tag{H1}$$

Then there exists a constant $\delta = \delta(p) > 0$ *such that* (1.3) *with* $b = p + \mu$ *has a solution* $N > 0$, $N \in \Omega$ *for* $0 < \mu < \delta$.

(b) *Let* $b \in B$, $[b] > 0$. *Assume the following:*

$$H_t N = c(t) N + G_t N,\ c \in B,\ c(t) > 0 \text{ where the operator } N \to G_t N \text{ satisfies (H1) with } \Omega = B \text{ (except that } [G_t N] = 0 \text{ is allowed) and maps nonnegative functions to nonnegative functions}. \tag{H2}$$

Then (1.3) *has a solution* $N > 0$, $N \in B$.

In particular, note that for the differential equation (1.1) one finds from part (b) of this theorem that a positive solution exists in B for any b and c in B provided that $[b] > 0$ and $c(t) > 0$ for all t. The integral in (1.2) satisfies (H1) if $\alpha(t, s) \in B$ for every s and is nondecreasing in s with $\int_0^\infty d\alpha(t, s) < +\infty$ for every t. If $\beta(t, s)$ satisfies these same conditions and if $\alpha(t, s) = c(t)\, u_0(s) + \beta(t, s)$ for $c \in B$, $c(t) > 0$, then (H2) is satisfied.

Before the stability of positive solutions $N \in B$ is considered, some simple observations concerning a certain weighted average of such solutions will be made. If Eq. (1.3) is divided by N and integrated from 0 to ω and the result is divided by ω, then one finds quite easily that $[H_t N] = [b]$. This is valid for any positive, bounded solution in Ω of (1.3). In particular, one has for Eq. (1.1) that the weighted average of any positive solution is $[cN] = [b]$. For H_t satisfying (H2) one has $[b] \geqslant [cN] \geqslant c_0[N]$ for any constant $0 < c_0 \leqslant c(t)$ and $N > 0$ satisfying (1.3).

THEOREM 2. *Suppose that* $b(t)$, $c(t) \in B$ *in* (1.1) *and that* $c(t) > 0$ *for all* t. *Then any positive solution* $\bar{N}$ *which is bounded away from zero is locally uniformly asymptotically stable. Moreover, for every* t_0 *there corresponds a constant* $\delta = \delta(t_0, b, c) > 0$ *such that if* $N(t; t_0, N_0)$ *is the solution of* (1.1) *with initial value* N_0 *at time* t_0 *and if* $N_0 > \bar{N}(t_0) - \delta$ *then* $N(t; t_0, N_0)$ *exists for all* t *and*

$$N(t; t_0, N_0) - \bar{N}(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Here uniform asymptotic stability means that which is commonly defined in the theory of ordinary differential equations [3]. The techniques used in proving this theorem utilize standard linearization techniques for ordinary differential equations. This concept of uniform asymptotic stability and these techniques are also available [3] for the functional differential equation (1.3) (see [1, especially Remark 1]). This approach results in our next theorem.

Let C_0 denote the Banach space of functions continuous for all t under the norm $|\cdot|_0$. Given a bounded linear operator $L: C_0 \to C_0$ let $|L| = \sup\{|LN|_0: |N|_0 = 1\}$.

$$\text{The operator } N \to H_t N \text{ satisfies (H2) where } G_t: C_0 \to C_0 \text{ is a bounded linear operator.} \tag{H3}$$

THEOREM 3. *Let* (H3) *hold. There exists a constant* $\gamma > 0$ *such that if* N *is any positive solution of* (1.3) *which is bounded away from zero and which satisfies* $|N|_0 \, |G_t| \leqslant \gamma$ *then* N *is* (*locally*) *uniformly asymptotically stable.*

These theorems obviously apply to positive solutions in B (e.g., those whose existence is guaranteed by Theorem 1) since positive periodic solutions are necessarily bounded away from zero.

Theorem 3 obtains stability at the expense of requiring that the lag or hereditary influence is small. That one needs some such assumption is evidenced by the work in [2, 4–6], where it is shown that significant lag or hereditary effects can create periodic solutions even in the presence of an equilibrium, the latter being not asymptotically stable.

The next and final result follows from Theorems 1 and 3.

COROLLARY. *Under the hypotheses of Theorem* 3, *there exists a constant* $\delta = \delta(b - [b]) > 0$ *such that if* $0 < [b] < \delta$ *and if* N *is the positive solution of* (1.3) *in* B *whose existence is guaranteed by Theorem* 1, *then* N *is* (*locally*) *uniformly asymptotically stable.*

The point of the corollary is that it turns out (see the proof of Theorem 1 in Section 3) that as $[b] \to 0$ in Theorem 1 the solution N is such that $|N|_0 \to 0$ so that the condition $|N|_0 \, |G_t| \leqslant \gamma$ in Theorem 3 can be met for $[b]$ sufficiently small.

3. PROOFS

In preparation for the proofs of the theorems in Section 2 some preliminary lemmas will be given.

LEMMA 1. *Suppose* H_t *satisfies* (H1) *and* $b(t) \in B$. *If* $N \in B$ *is a solution of* (1.3), *then either* $N(t) < 0$, $N(t) > 0$, *or* $N(t) \equiv 0$ *for all* t.

Proof. If, for a given solution N, one defines the continuous function $g(t) \equiv b(t) - H_t N$ then $N(t)$ solves the linear equation $x' = g(t)\, x$. Thus, $N = N(t_1) \exp(\int_{t_1}^{t} g(s)\, ds)$ and the result follows. ∎

LEMMA 2. *Suppose $a(t) \in B$. If $[a] < 0$, then the nonhomogeneous equation*

$$x' = a(t)\,x + f(t) \tag{3.1}$$

has a unique solution $x \in B$ for each $f \in B$ and the map $L\colon B \to B$ defined by $x = Lf$ is linear and compact.

Proof. This result follows from a well-known theorem [3, p. 225] since the assumption $[a] < 0$ guarantees that the homogeneous equation has no solution in B. ▮

LEMMA 3. *Suppose that $a(t), f(t) \in B$ with $[a] = 0$. Then all solutions of $y' = a(t)\,y$ lie in B and in order for (3.1) to have a solution in B it is necessary that $[fy^*] = 0$ where y^* is any solution of $y' = -a(t)\,y$.*

Proof. First observe that $\int_0^t a(s)\,ds \in B$ follows from the assumption that $[a] = 0$. This implies $\exp(\pm \int_0^t a(s)\,ds) \in B$ and hence that all solutions of both equations $y' = \pm a(t)\,y$ lie in B. The orthogonality condition follows easily by integrating y^*x' by parts. ▮

Proof of Theorem 1. (a) Let $\lambda = \mu + 1$ and $a(t) = p(t) - 1 \in B$. Then (1.3) becomes

$$N' = (\lambda + a(t))\,N - NH_tN.$$

Observing that $[a] = -1 < 0$ and referring to Lemma 2 one finds that as far as positive solutions of (1.3) in Ω are concerned this equation is equivalent to the operator equation $N = L(\lambda N - NH_t \mid N \mid)$ or

$$N = \lambda LN + H(N), \qquad N \in B, \tag{3.2}$$

where by (H1) the operator $H\colon \Omega \to B$ is continuous and satisfies $H(N) = -LNH_t \mid N \mid = o(\mid N \mid_0)$ near $N = 0$. By Lemma 2 and (H1), H is completely continuous. Using λ as a free nonlinear eigenvalue, one may apply well-known bifurcation techniques [8] to obtain nontrivial solutions of (3.2). This means that one must investigate the linearized problem $N = \lambda LN$, $N \in B$. A characteristic solution (λ, N) of L in $R \times B$ yields a nontrivial solution of the scalar differential equation (and vice versa)

$$N' = (\lambda + a(t))\,N. \tag{3.3}$$

Such a solution exists if and only if $[\lambda + a] = 0$, i.e., if and only if $\lambda = -[a] = 1$. Thus, there exists one and only one characteristic value of L on B (namely, $\lambda = 1$) which, since (3.3) is scalar, is simple (of multiplicity 1). Applying [8, Theorem 1.25], one obtains the existence of two (locally distinct) continua $C_\pm$ of nontrivial solutions of (3.2) bifurcating from (1, 0) in $R \times B$ (i.e., whose closures contain (1, 0)).

By Lemma 1 any continuum of nontrivial solutions must consist of solutions of one sign. Since $H(N)$ is odd (i.e., $H(-N) = -H(N)$) it follows that $-N$ is a solution of (3.2) whenever $N \in \Omega$ is a sufficiently small solution (so that $-N \in \Omega$). Consequently one of the two continua (say C_+) consists of positive solutions. Any solution $(\lambda, N) \in C_+$ of (3.2) then yields a positive solution $N \in \Omega$ of (1.3) for $\mu = \lambda - 1$ and small. According to the remarks immediately preceding Theorem 2, $\mu = [H_t N]$. Thus, by (H1) one has that $\mu > 0$ for such solutions and part (a) is proved.

(b) It follows from [8, Theorem 1.25] that C_+ connects to the boundary of Ω. Under the assumption (H2) this means C_+ connects to ∞, i.e., that C_+ is unbounded in $R \times B$. Since C_+ is a continuum it follows that the image of at least one of the projections onto either R or B is unbounded. It is shown next that (H2) implies that the image I of the projection onto R is $I = (1, +\infty)$.

First observe that since C_+ bifurcates from $(1, 0) \in R \times B$, it follows that $\lambda = 1$ is contained in the closure of I. Next observe that for every $(\lambda, N) \in C_+$ one has $\lambda - 1 = \mu = [H_t N] > 0$ by (H2) and hence $I \subseteq (1, +\infty)$. Equality will follow from the proof that I is unbounded.

First an a priori estimate on N in terms of λ for $(\lambda, N) \in C_+$ is obtained by use of (H2). For such a function

$$N' = (\lambda + a(t))\, N - c(t)\, N^2 - NG_t N.$$

Let t_m be a point where $N(t_m) = | N |_0$; then necessarily $N'(t_m) = 0$ so that $0 \leqslant (\lambda + a(t_m))\, | N |_0 - c(t_m)\, | N |_0^2$, and hence,

$$| N |_0 \leqslant (\lambda + a(t_m))/c(t_m) \leqslant (\lambda + | a |_0)/c_0, \tag{3.4}$$

where $c_0 > 0$ is a constant such that $0 < c_0 \leqslant c(t)$.

If I were bounded it would follow from (3.4) that the image of the projection of C_+ onto B is bounded, a contradiction. Thus, it has been shown that I is unbounded, but cannot be unbounded below. Since C_+ is a continuum in $R \times B$ it certainly is true that I is a continuum in R and consequently is an interval, unbounded above; thus, $I = (1, +\infty)$.

The proof of part (b) of Theorem 1 follows from the observation that for any average $[b] > 0$ there exists a $\lambda \in I$ such that $\lambda = [b] + 1$. ∎

Proof of Theorem 2. Let $\bar{N}$ be a fixed, positive solution of (1.1) which satisfies $\bar{N} \geqslant \alpha > 0$ for all t and some constant $\alpha > 0$. Define $x = N - \bar{N}$; then x satisfies the equation

$$x' = (\bar{N}'/\bar{N} - c\bar{N})\, x - cx^2. \tag{3.5}$$

The associated linear equation

$$y' = (\bar{N}'/\bar{N} - c\bar{N})\, y \tag{3.6}$$

has fundamental solution $y = \bar{N}(t)\exp(-\int_0^t c\bar{N}\, du)$. Thus,

$$|\, y(t)\, y^{-1}(s)| = |\, \bar{N}(t)\, \bar{N}^{-1}(s)| \exp\left(-\int_s^t c\bar{N}\, du\right) \leqslant |\, \bar{N}\,|_0\, \alpha^{-1} \exp(-c_0\alpha(t-s)) \tag{3.7}$$

for all $t \geqslant s$ and it follows from standard linearization theorems [3, p. 63] that $x = 0$ (i.e., $N = \bar{N}$) is (locally) uniformly asymptotically stable. Hence, $|\, \bar{N}(t_0) - N_0\,| \leqslant \delta$ implies $N(t; t_0, N_0) - \bar{N}(t) \to 0$ as $t \to +\infty$ for some constant $\delta > 0$.

The variation of constants formula implies

$$x(t) = y(t)\, y^{-1}(t_0)\, x(t_0) - \int_{t_0}^t y(t)\, y^{-1}(s)\, c(s)\, x^2(s)\, ds. \tag{3.8}$$

Suppose $N_0 \geqslant \bar{N}(t_0)$ so that $x(t_0) \geqslant 0$. The uniqueness of solutions of (3.5) implies that $x(t) \geqslant 0$ for all t. From (3.8) one finds that $0 \leqslant x(t) \leqslant y(t)\, y^{-1}(t_0) \times x(t_0)$, which implies, together with (3.7), that x exists for all t and $x(t) \to 0$ as $t \to +\infty$. ∎

Proof of Theorem 3. Let $x = N - \bar{N}$ where $\bar{N} > 0$ solves (1.3). Then x satisfies the equation

$$x' = (\bar{N}'/\bar{N} - c\bar{N})\, x - \bar{N}G_t x - cx^2 - xG_t x. \tag{3.9}$$

The linear ordinary differential equation (3.6) is exponentially stable, as is seen by (3.7). Converting (3.9) to an integral equation by integrating both sides from 0 to t, we find that Theorem 3 follows immediately from [1, Theorem 1(ii) and Remark 1]. ∎

Proof of the Corollary. From the proof of Theorem 1, part (a) one finds that because the solutions $N \in B$ branch from the trivial solution $N \equiv 0$ for λ near 1 (i.e., as $\lambda \to 1$ the solutions satisfy $|\, N\,|_0 \to 0$), it follows that given $\epsilon > 0$, there exists a constant $\delta > 0$ such that for $0 < [b] < \delta$ one has $|\, N\,|_0 \leqslant \epsilon$ for the solution N on this branch. In particular, one may take $\epsilon = \gamma\,|\, G_t\,|^{-1}$. Theorem 3 then applies. ∎

References

1. J. M. Bownds and J. M. Cushing, On preserving stability of Volterra integral equations under a general class of perturbations, *Math. Systems Theory* **9**, No. 2 (1975), 117–131.

2. J. M. CUSHING, Periodic solutions of Volterra's population equation with hereditary effects, *SIAM J. Appl. Math.* **31**, No. 2 (1976), 251–261.*
3. A. HALANAY, "Differential Equations: Stability, Oscillations, Time Lags," Academic Press, New York, 1966.
4. G. S. JONES, On the nonlinear differential-difference equation $f'(x) = -\alpha f(x-1) \times [1 + f(x)]$, *J. Math. Anal. Appl.* **3** (1962), 440–469.
5. G. S. JONES, The existence of periodic solutions of $f'(x) = -\alpha f(x-1)[1 + f(x)]$, *J. Math. Anal. Appl.* **5** (1962), 435–450.
6. R. M. MAY, "Stability and Complexity in Model Ecosystems," Monographs in Population Biology, No. 6, Princeton Univ. Press, Princeton, N.J., 1974.
7. R. K. MILLER, On Volterra's population equation, *SIAM J. Appl. Math.* **14**, No. 3 (1966), 446–452.
8. P. H. RABINOWITZ, Some global results for nonlinear eigenvalue problems, *J. Functional Anal.* **7** (1971), 487–513.
9. V. VOLTERRA, "Leçons sur la Théorie Mathématique de la Lutte pour la Vie," Gauthier-Villars, Paris, 1931.

* Also see J. M. CUSHING, Errata to "Periodic solutions of Volterra's population equation with hereditary effects," *SIAM J. Appl. Math.* **32**, No. 4 (1977), 895.