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A Uniqueness Criterion for Harmonic Functions Under Nonlinear Boundary Conditions

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1. INTRODUCTION

Consider the problem

$$\begin{gathered} \Delta u \equiv u_{xx} + u_{yy} = 0 \quad \text{in} \quad D \supset R, \\ u = 0 \quad \text{on} \quad C_1, \quad \frac{\partial u}{\partial n} = h(s)f(u) \quad \text{on} \quad C_2 \end{gathered} \tag{1.1}$$

where R, D are a regular regions [10] in x, y space and where the boundary $\partial R = C_1 + C_2$. Here s is arc length along ∂R, $h(s)$ is an integrable function of s defined on C_2, f is a prescribed function of u such that $f(0) = 0$ (more will be assumed about f below), and $\partial u/\partial n$ is the derivative of u in the direction of the outward normal to C_2. By a solution to this problem we shall mean a function $u \in C^2(D)$ which satisfies (1.1).

It is well known [3] that solutions to (1.1) may *not* be unique, even for the linear problem $f(u) \equiv u$. For example, if R is the unit disk, $C_2 = \partial R$, and $h(s) \equiv m =$ const., then (1.1) has solutions in polar coordinates r, θ given by $r^m(k_1 \sin m\theta + k_2 \cos m\theta)$ for $m =$ positive integer and k_1, k_2 equal to any constants; moreover, these are the only solutions for a given m [3]. Notice, however, that there is at most one solution (up to a constant multiple) which possess a given set of nodal lines. Martin [6] has extended this remark to more general linear problems by showing that if $f \equiv u$ in (1.1) then there cannot exist two nonconstant linearly independent solutions u_1, u_2 for which the ratio u_1/u_2 remains analytic in D. Martin remarks further (without proof) that this condition on u_1, u_2 is equivalent to requiring that u_1 have a nodal line wherever u_2 does. Similar results have been derived concerning various types of uniqueness for the nonlinear problem (1.1) under suitable restrictions on f provided $\lambda = f(u_2)/f(u_1)$ remains analytic in R (cf. Martin [6, 7, 8, 9], Dunninger [4, 5], Cushing [1, 2]). The conditions that λ remain analytic is the nonlinear analog of Martin's theorem for the linear problem and may be interpreted in terms of equipotential lines of u_1 (see Lemma 3.2 below).

This suggests we formulate uniqueness questions for (1.1) in terms of equivalence classes of the set of harmonic functions on D where $u_1 \sim u_2$ if and only if the nodal lines $u_1 = 0$, $u_2 = 0$ coincide (we assume in Theorem 2.1 that $u = 0$ is the *only* zero of $f(u)$). We denote by $E(u_1)$ the equivalence class of u_1 under this equivalence relation. In Lemmas 3.1, 3.2 we show explicitly the relationship between $E(u_1)$ and the ratio λ under certain conditions of f. Lemma 3.2 allows certain theorems of Martin, Dunninger and the author to be stated as uniqueness theorems within equivalence classes; e.g., if $f \equiv u$, Martin's theorem states that two solutions belonging to the same equivalence class are linearly dependent. Our main purpose in this paper is to prove for the nonlinear problem (1.1) an analog of Martin's result for the linear problem by showing the uniqueness (up to a sign) within equivalence classes of nonconstant solutions to (1.1) provided f is an odd, monotonic function of u possessing an inflection point at $u = 0$ of a definite type.

2. Results

The following theorem contains our main result.

THEOREM 2.1. *Suppose $f = f(u)$ satisfies the following conditions as a function of u:*

$$
\begin{aligned}
&\text{(a)}\quad f \textit{ is } n+2 \textit{ times continuously differentiable for for some } n \geqslant 1;\\
&\text{(b)}\quad f^{(k)}(u) \equiv d^k f/du^k \neq 0 \textit{ at } u = 0 \textit{ for some } 1 \leqslant k \leqslant n;\\
&\text{(c)}\quad f(u) = -f(-u) \textit{ for all } u;\\
&\text{(d)}\quad f^{(1)}(u) > 0 \textit{ for all } u \neq 0;\\
&\text{(e)}\quad f^{(2)}(u) < 0 \textit{ for all } u > 0.
\end{aligned}
\tag{2.1}
$$

If u_1, u_2 are nonconstant solutions to (1.1) *belonging to the same equivalence class, then $u_2 \equiv \pm u_1$ on R.*

The theorem is proved by a sequence of lemmas given in Sec. 3. Lemma 3.2 implies that for two solutions u_1, u_2 satisfying $u_2 \in E(u_1)$ we have λ, λ^{-1} both C^1 in D. Two applications of Lemma 3.3 (obtained by interchanging the roles of u_1 and u_2) yield the inequalities $|u_1| \leqslant |u_2|$ and $|u_2| \leqslant |u_1|$; thus, $|u_1| \equiv |u_2|$ and the theorem follows.

As an example, this theorem applies to the problem obtained from $f \equiv \sin u$ (at least for solutions satisfying $-\pi/2 < u < \pi/2$) studied by Martin in [8, 9] (and Dunninger in [4]). This result also bears an interesting relationship to the local uniqueness theorems of the author in [1] where it is assumed that $f(u)f^{(2)}(u) \leqslant 0$ for all u.

Notice, finally, that for problems (1.1) with $f^{(1)}(u) < 0$, $u \neq 0$, we may replace $h(s)$ by $-h(s)$ and $f(u)$ by $-f(u)$ and apply Theorem 2.1.

3. Three Lemmas

It is clear that if the ratio u_1/u_2 of two harmonic functions is an analytic function in D, then the nodal lines of u_2 must coincide with nodal lines of u_1. The converse of this statement (which Martin mentions in [6] without proof) is not immediately obvious; therefore, we offer a proof.

Lemma 3.1. *Let u_1, u_2 be two nonconstant functions which are harmonic in an open region D. Then u_1/u_2 is analytic in D if and only if each nodal line of u_2 in D coincides with a nodal line of u_1 in D. Thus, both u_2/u_1 and u_1/u_2 are analytic in D if and only if $u_2 \in E(u_1)$.*

We have only to prove the converse. Certainly u_1/u_2 is analytic at those points in D where $u_2 \neq 0$. In order to consider points $(x_0, y_0) \in D$ for which $u_2 = 0$ we develop a canonical representation for a harmonic function u in the neighborhood of this point. The family of nodal lines passing through (x_0, y_0) consists of a finite number (say $n \geqslant 1$) of analytic curves whose slopes are spaced $2\pi/n$ radians apart (cf. Walsh [11]). Assume without loss of generality that $x_0 = y_0 = 0$ and that none of the nodal lines has a vertical slope; this can always be achieved by a translation and/or rotation of coordinate axes. Then the nodal lines may be represented by $y = g_i(x)$ ($i = 1, 2, \ldots, n$) where g_i is an analytic function of x, and we may write

$$u(x, y) = U(x, y) \prod_{i=1}^{n} [y - g_i(x)], \tag{3.1}$$

where U is analytic at (0, 0). To see this, let $\xi = y - g_1(x)$, $\eta = x$; under this proper change of variables u becomes an analytic function of ξ, η which vanishes for $\xi = 0$ and, hence, $u = \xi U^*$ where $U^* = U^*(\xi, \eta)$ is analytic. Consequently, $u = [y - g_1(x)]\, U_1(x, y)$ where U_1 is analytic and vanishes for $y = g_2(x)$; n repetitions of this argument clearly leads to (3.1). Moreover, if v is the harmonic conjugate of u such that $v(0, 0) = 0$, then $u + iv = z^n h(z)$, $z = x + iy$, where $h(z)$ is analytic and $h(0) \neq 0$ (Walsh [11], pg. 269) and it follows that the lowest order terms appearing in the power series development of u are of order n. This implies $U(0, 0) \neq 0$ in (3.1). Applying the decomposition (3.1) to u_1, u_2 satisfying the hypotheses of the theorem we get

$$u_1 = U_1 \prod_{i=1}^{n} (y - g_i), \qquad u_2 = U_2 \prod_{i=1}^{m} (y - g_i)$$

where $U_1(0, 0) \neq 0$, $U_2(0, 0) \neq 0$ and $n \geqslant m$. Thus,

$$\frac{u_1}{u_2} = \prod_{i=m+1}^{n} (y - g_i) \frac{U_1}{U_2}$$

is analytic at $x = y = 0$. The second statement of the lemma is an immediate consequence of the first.

LEMMA 3.2. *If $f(u)$ is $n + 2$ times continuously differentiable as a function of u satisfying $f^{(k)}(0) = 0$, $0 \leqslant k \leqslant n - 1$, $f^{(n)}(0) \neq 0$ where $n \geqslant 1$ and if $u = 0$ is the only zero of f on the range of two harmonic functions u_1, u_2 on D, then $\lambda \equiv f(u_1)/f(u_2)$ is C^1 as a function of x, y in D provided u_1/u_2 is analytic in D. Thus, $u_2 \in E(u_1)$ if and only if both λ and λ^{-1} are C^1 in D.*

This follows immediately from the preceding lemma and the expression

$$\lambda = \left(\frac{u_2}{u_1}\right)^n \frac{f^{(n)}(0) + R(u_2)}{f^{(n)}(0) + R(u_1)}$$

where R is the remainder term in Taylor's expansion of $f(u)$.

LEMMA 3.3. *Let $f(u)$ satisfy the hypotheses of Theorem 2.1. If u_1, u_2 are nonconstant solutions to (1.1) such that $\lambda \equiv f(u_1)/f(u_2)$ is C^1 in D, then $|u_1| \leqslant |u_2|$ on D.*

To prove this lemma we begin with the integral identity

$$\int_{\partial S} \lambda \left(f_2 \frac{\partial u_1}{\partial n} - f_1 \frac{\partial u_2}{\partial n}\right) ds \equiv B = A \equiv \int_S (Q + \lambda f_2 \Delta u_1 - \lambda f_1 \Delta u_2)\, dx\, dy, \tag{3.2}$$

which is a special case of a generalized Green's identity introduced by Martin in [6]. Here we have set

$$Q \equiv f_1^{(1)} p_1^2 - 2\lambda f_1^{(1)} p_1 p_2 + \lambda^2 f_2^{(1)} p_2^2 + f_1^{(1)} q_1^2 - 2\lambda f_1^{(1)} q_1 q_2 + \lambda^2 f_2^{(1)} q_2^2, \tag{3.3}$$

where

$$p_i = \frac{\partial u_i}{\partial x}, \qquad q_i = \frac{\partial u_i}{\partial y}, \qquad f_i = f(u_i),$$

and

$$f_i^{(1)} = \frac{df(u_1)}{du} \qquad (i = 1, 2).$$

This identity is a straight forward application of the divergence theorem provided the divergence theorem is valid on S and λ is C^1 in $S + \partial S$. Treating

Q as a quadratic form in p_i, q_i with continuous coefficients, one can show without difficulty (by examining the descending principal minors) that Q is positive definite if and only if

$$\lambda^2(f_1^{(1)} - f_2^{(1)}) f_1^{(1)} < 0 \qquad \text{on} \qquad S, \tag{3.4}$$

$$f_1^{(1)} > 0 \qquad \text{on} \qquad S. \tag{3.5}$$

Condition (3.5) holds because of (2.1d). Assume $|u_1| > |u_2|$ at some point $(x_0, y_0) \in D$; we now search for a subregion S of D on which (3.4) holds.

Since condition (2.1c) implies u_1, u_2 are solutions to (1.1) if and only if $-u_1$, $-u_2$ are also solutions, we may assume without loss of generality that $u_1 > u_2 > 0$ at (x_0, y_0); consequently, $S = \{(x, y) \in D : u_1 > u_2 > 0\}$ is a non-empty, open subset of D. The boundary ∂S consists of arcs Γ_1 on ∂R, arcs Γ_2 on the (analytic) nodal lines $u_2 - u_1 = 0$ $(u_2 \neq 0)$, and/or arcs Γ_3 on the (analytic) nodal lines $u_1 = u_2 = 0$ $(\lambda \in C^1 \Rightarrow u_2 \in E(u_1))$ and, hence, S is a regular subregion [10] of R over which the divergence theorem is valid [10]. Thus, (3.2) is valid on S. Since condition (3.4) also holds on S, Q is positive definite and $A \geqslant 0$. Clearly, for two solutions u_1, u_2 to (1.1) the integrand of B vanishes on Γ_1, Γ_3 and as a result

$$B \equiv \int_{\Gamma_2} f_1 \frac{\partial(u_1 - u_2)}{\partial n}\, ds \leqslant 0,$$

since $f_1 > 0$ on S and $\partial(u_1 - u_2)/\partial n \leqslant 0$ where n is the outwardly directed normal on Γ_2. Thus, (3.2) implies $A = B = 0$ and the definiteness of Q implies the contradiction that u_1, u_2 are constant in S (and, hence, R). We conclude that no point exists in D for which $|u_1| > |u_2|$; i.e., $|u_1| \leqslant |u_2|$ on D and the lemma is proved.

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