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# UNIQUENESS OF POSITIVE SOLUTIONS TO NONLINEAR ELLIPTIC PROBLEMS*

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**Abstract.** The uniqueness of positive solutions to self-adjoint elliptic partial differential equations with nonlinear forcing terms subject to mixed Dirichlet and nonlinear Neumann boundary conditions on bounded domains is proved under relatively mild conditions on the nonlinear terms. The result generalizes known results.

**1. Introduction.** Recently, positive solutions to certain nonlinear elliptic partial differential equations have been of interest (cf. [1], [2], [4], [5]). The uniqueness of such solutions is known for nonlinear elliptic problems with linear boundary conditions under the assumption that the nonlinear terms are of a restrictive form and satisfy a concavity condition (see [1], [2], [4], [5]). In Theorem 1 below we generalize these known results in several directions: first, we consider a nonlinear differential equation of a general type (see (2.1) below); secondly we consider nonlinear boundary conditions (see (2.2) below); and, finally, we weaken the assumptions on the nonlinear terms (H1–H3 below).

**2. Results.** Consider the following general boundary value problem which will be referred to as Problem I:

$$Lu = F(x, u) \quad \text{on } D, \tag{2.1}$$

$$u(x) = \alpha(x) \quad \text{on } S^1, \qquad \partial u/\partial v = G(x, u) \quad \text{on } S^2, \quad S^1 + S^2 = S, \tag{2.2}$$

where $x = (x_1, \cdots, x_m)$ and

$$Lu \equiv \sum_{i,j=1}^{m} D_i(a_{ij}(x)D_ju) + a_0(x)u, \qquad D_i = \partial/\partial x_i,$$

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j > 0, \qquad \xi_1^2 + \cdots + \xi_m^2 \neq 0, \quad x \in D,$$

$$a_{ij}(x) = a_{ji}(x), \qquad x \in D,$$

$$\frac{\partial u}{\partial v} \equiv \sum_{i,j=1}^{m} a_{ij}(x)n_i(x)D_ju, \qquad x \in S.$$

Here $D$ is a bounded region in $m$-dimensional space with boundary $S$ whose outwardly directed normal at $x$ is denoted by $(n_1(x), \cdots, n_m(x))$; $S^1$ and $S^2$ are disjoint measurable sets whose union is $S$. (Actually our proof and hence our result are valid when $S^1$ and $S^2$ are disjoint, measurable sets whose union equals $S$ up to a set of measure zero, but we will not push this point.) The divergence theorem is assumed to hold on $D$ and the coefficients $a_{ij}(x)$, $a_0(x)$ are assumed once continuously differentiable on $\bar{D}$, the closure of $D$. The functions $\alpha$, $F$, $G$ are presumed given in advance. By a solution to Problem I we mean a function

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$u(x) \in C^1(\bar{D})$ for which the derivatives appearing in (2.1) exist and are continuous on $\bar{D}$, and the boundary conditions (2.2) hold on the appropriate regions.

We impose the following conditions on the given functions $\alpha$, $F$, $G$:

H1. $\alpha(x)$, $F(x,z)$, $G(x,z)$ are all defined and continuous on $S^1$, $D \times [0,\infty)$, $S^2 \times [0,\infty)$, respectively, and $\alpha(x) > 0$ on $S^1$.

H2. $z'F(x,z) \geqq zF(x,z')$ for $z \geqq z' \geqq 0$ and $x \in D$.

H3. $z'G(x,z) \leqq zG(x,z')$ for $z \geqq z' \geqq 0$ and $x \in S^2$.

Our main result, which is proved in the next section, is contained in the following theorem.

THEOREM 1. *If* H1, H2, H3 *hold and if $u$, $v$ are two solutions to Problem* I *satisfying $u > 0$, $v > 0$ on $\bar{D}$, then $u = kv$, $k$ = positive constant. Consequently, if $S^1$ is nonempty, there exists at most one positive solution to Problem* I. *In any case if a strict inequality holds in either* H2 *or* H3, *then at most one positive solution exists.*

Hypothesis H2 (H3) means geometrically that the slope of the line in the $z, F$-plane ($z, G$-plane) passing through the origin and the "point" $[z, F(x,z)]$ ($[z, G(x,z)]$) is a nondecreasing (nonincreasing) function of $z \geqq 0$ for each fixed value of $x$. If $F$, $G$ are once differentiable in $z$ for all values of $x$ in the appropriate regions, then H2, H3 are equivalent to the requirements $zF_z - F \geqq 0$, $zG_z - G \leqq 0$ for all $z \geqq 0$ and appropriate $x$ (see [2]). It is not difficult to see that any functions $F$, $G$ which are concave up and concave down in $z$, respectively, and which satisfy $F(x,0) \leqq 0$, $G(x,0) \geqq 0$ for all appropriate $x$ necessarily satisfy H2, H3, respectively. Moreover, H2 and H3 are certainly satisfied for functions $F$, $G$ linear in $z$ and, thus, these hypotheses (which do not necessarily restrict the concavity or monotonicity of $F$, $G$ in the variable $z$) are weaker than the concavity assumption of Keller [4], [5] and Cohen [1], [2] for Problem I.

**3. Proof of Theorem 1.** The proof utilizes a generalization of Green's integral identity (due originally to M. H. Martin) which has been used by many authors to study uniqueness questions for nonlinear boundary problems (cf. Cushing [3] for bibliography). A straightforward application of the divergence theorem together with $a_{ij} = a_{ji}$ yields the identity

$$\int_S (\lambda - 1)\left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) dx = \int_D [Q + (\lambda - 1)(vLu - uLv)]\, dx, \tag{3.1}$$

where $\lambda = u/v$ and $Q = v^2 \sum_{i,j=1}^m a_{ij} D_i\lambda D_j\lambda$. Supposing that $u$, $v$ are two solutions to Problem I satisfying $u > 0$, $v > 0$ in $\bar{D}$, we see that this identity becomes

$$I_1 \equiv \int_{S^2} (\lambda - 1)[vG(x,u) - uG(x,v)]\, dx = I_2 + I_3, \tag{3.2}$$

where

$$I_2 \equiv \int_D Q\, dx, \qquad I_3 \equiv \int_D (\lambda - 1)[vF(x,u) - uF(x,v)]\, dx.$$

As $v > 0$ on $\bar{D}$, we have $\lambda \in C^1(\bar{D})$ and consequently the identity is valid. Now $I_2 \geqq 0$ by the definiteness of $a_{ij}$; moreover, the integrand of $I_3$ is nonnegative (H2) while the integrand of $I_1$ is nonpositive (H3), and hence $I_1 \leqq 0$, $I_3 \geqq 0$. We conclude from identity (3.2) that $I_i = 0$, $i = 1, 2, 3$. But $I_2 = 0$ together with the definiteness of $Q$ implies $D_i\lambda = 0$, $i = 1, \cdots, m$, or $u = kv$, $k =$ const. If $S^1$ is nonempty, then clearly $k = 1$. In any case, if strict inequality holds in H2 or H3, then $I_3 = 0$ or $I_1 = 0$ implies $k = 1$ and the theorem follows.

Finally we note that for eigenvalue problems of the general type $F \equiv F(\lambda, x, u)$ and/or $G \equiv G(\mu, x, u)$, $\lambda$, $\mu =$ constants (to be determined as part of the solution), Theorem 1 remains valid for all eigenvalues $\lambda$ and/or $\mu$ for which a positive solution exists provided H1–H3 hold for the given values of $\lambda$, $\mu$.

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