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Uniqueness Theorems under Nonlinear Boundary Conditions for Some Quasi-Linear Elliptic Equations

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1. INTRODUCTION

In [3-6] we considered the uniqueness of an analytic solution $w = u + iv$ to the nonlinear boundary problem

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial n} = f(u, v, s) \quad \text{on } \partial S, \tag{1.1}$$

where S is a simply connected region bounded by a simple smooth closed curve ∂S along which s denotes the arc length and $\partial u/\partial n$ denotes the external normal derivative of u on ∂S.

In as much as u and v satisfy the Cauchy-Riemann equations, we can rewrite (1.1) in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \quad \text{in } S, \qquad \frac{\partial u}{\partial n} = f(u, v, s) \qquad \text{on } \partial S, \tag{1.2}$$

and hence we are led to consider the uniqueness of a solution pair $[u, v]$ to the following generalized system

$$\left.\begin{aligned} \frac{\partial v}{\partial y} &= A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} &= -B\frac{\partial u}{\partial x} - C\frac{\partial u}{\partial y} \end{aligned}\right\} \quad \text{in } S, \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} = f(u, v, s) \quad \text{on } \partial S, \tag{1.4}$$

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or of the problem

$$Lu = (Au_x)_x + (Bu_x)_y + (Bu_y)_x + (Cu_y)_y = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = f(u, v, s) \quad \text{on } \partial S, \tag{1.5}$$

obtained by eliminating v from (1.3). Here, $\partial u/\partial \nu$ denotes the conormal derivative of u with respect to the operator L:

$$\frac{\partial u}{\partial \nu} = \nu_1 u_x + \nu_2 u_y ,$$

where

$$\nu_1 = An_1 + Bn_2 , \qquad \nu_2 = Bn_1 + Cn_2 ,$$

$n(x, y) = (n_1(x, y), n_2(x, y))$ being the external unit normal vector at a point (x, y) on ∂S, and A, B, C are single-valued continuously differentiable functions of x, y, u which satisfy the ellipticity condition $B^2 - AC < 0$ in S. (Without loss of generality, we may assume $A > 0$ in S). The functions u, v are assumed continuously differentiable in $S + \partial S$ and twice continuously differentiable in S.

2. The Basic Integral Identity

Suppose $[u_1 , v_1]$ and $[u_2 , v_2]$ are two solutions of (1.5). A straightforward application of Gauss' theorem together with (1.3) verifies the formal integral identity

$$\int_{\partial S} \tau \left(f_2 \frac{\partial u_1}{\partial \nu} - f_1 \frac{\partial u_2}{\partial \nu} \right) ds = \int_S Q \, dS, \tag{2.1}$$

where

$$Q = a_1 p_1^2 + 2b_1 p_1 p_2 + c_1 p_2^2 + 2d_1 p_2 q_1 + 2d_2 p_1 q_2 + 2e_1 p_1 q_1 + 2e_2 p_2 q_2 + a_2 q_1^2 + 2b_2 q_1 q_2 + c_2 q_2^2,$$

is a quadratic form in $p_i = \partial u_i/\partial x$, $q_i = \partial u_i/\partial y$, $(i = 1, 2)$, with coefficients

$$\begin{aligned}
a_1 &= A_1(\tau f_2)_{u_1} , \qquad c_1 = - A_2(\tau f_1)_{u_2} ,\\
2b_1 &= A_1(\tau f_2)_{u_2} - A_2(\tau f_1)_{u_1} + (A_2 B_1 - A_1 B_2) [(\tau f_1)_{v_1} + (\tau f_2)_{v_2}],\\
a_2 &= C_1(\tau f_2)_{u_1} , \qquad c_2 = - C_2(\tau f_1)_{u_2} ,\\
2b_2 &= C_1(\tau f_2)_{u_2} - C_2(\tau f_1)_{u_1} + (B_2 C_1 - B_1 C_2) [(\tau f_1)_{v_1} + (\tau f_2)_{v_2}],\\
2d_1 &= B_1(\tau f_2)_{u_2} - B_2(\tau f_1)_{u_1} + (A_2 C_1 - B_1 B_2) [(\tau f_1)_{v_1} + (\tau f_2)_{v_2}],\\
2d_2 &= B_1(\tau f_2)_{u_2} - B_2(\tau f_1)_{u_1} - (A_1 C_2 - B_1 B_2) [(\tau f_1)_{v_1} + (\tau f_2)_{v_2}],\\
e_1 &= B_1(\tau f_2)_{u_1} , \qquad e_2 = - B_2(\tau f_1)_{u_2} ,
\end{aligned} \tag{2.2}$$

in which the function $\tau = \tau(u_1, u_2, v_1, v_2)$ is at our disposal and

$$f_1 = f_1(u_1, v_1), \qquad f_2 = f_2(u_2, v_2),$$

are either specified in advance arbitrarily, or are related to the boundary problem (1.5) by setting

$$f(u_1, v_1, s) = h(s) f_1(u_1, v_1), \qquad f(u_2, v_2, s) = h(s) f_2(u_2, v_2).$$

Finally, A_i, B_i, C_i are the coefficients in (1.5) evaluated at $[u_i, v_i]$, $(i = 1, 2)$.

The identity (2.1), which is valid for two solutions $[u_1, v_1]$, $[u_2, v_2]$ of (1.5) and for arbitrary functions f_1, f_2, τ so long as the integrals actually exist, is the source of our uniqueness theorems.

3. Uniqueness Theorems for the General Problem

Setting

$$\tau = \tau(u, v), \qquad [u, v] = [u_1, v_1], \qquad [u_2, v_2] = [0, 0], \qquad f_1 = f_2 = 1,$$

in (2.1) we obtain the integral identity

$$\int_{\partial S} \tau(u, v) \frac{\partial u}{\partial \nu} ds = \int_S \tau_u(u, v) (Ap^2 + 2Bpq + Cq^2) \, dS, \tag{3.1}$$

valid for any solution $[u, v]$ of (1.5) and any continuously differentiable function τ, from which we can easily deduce some uniqueness theorems. For the special case $A = C = 1$, $B = 0$, the identity (3.1) is due to Cushing [3].

Theorem 3.1. *If $[u, v]$ is a solution of the boundary problem*

$$Lu = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial S,$$

then $[u, v]$ is constant, i.e., both u and v are constant.

By setting $\tau = u$ in (3.1) we see that

$$\int_S (Ap^2 + 2Bpq + Cq^2) \, dS = 0,$$

and hence $p = q = 0$ or $u =$ constant since the ellipticity condition $B^2 - AC < 0$, $A > 0$ implies that the quadratic form $Ap^2 + 2Bqp + Cq^2$ is positive definite. The fact that v is constant readily follows from (1.3).

THEOREM 3.2. *If u is a solution of the differential equation $Lu = 0$ in S and v is such that $[u, v]$ is a solution pair of* (1.3), *then*

$$\int_{\partial S} \tau(v) \frac{\partial u}{\partial \nu}\, ds = 0, \tag{3.2}$$

for any continuously differentiable function τ.

Note that by setting $A = C = 1$, $B = 0$, the differential equation $Lu = 0$ becomes Laplace's equation and that by further setting $\tau = 1$ in (3.2) we obtain the well known theorem of Gauss [2].

COROLLARY 3.1. *The boundary problem*

$$Lu = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = G(u, s) f(v) \quad \text{on } \partial S, \tag{3.3}$$

where $f(v)$ and $G(u, s)$ are continuously differentiable functions can have no non-constant solution $[u, v]$ such that $G(u, s) \neq 0$ on ∂S.

Setting $\tau = f(v)$ in (3.2) yields, for any solution $[u, v]$ of (3.3),

$$\int_{\partial S} G(u, s) f^2(v)\, ds = 0,$$

and hence $f(v) \equiv 0$ on ∂S if $G(u, s) \neq 0$ on ∂S. This implies $\partial u/\partial \nu = 0$ on ∂S, and by Theorem 3.1 the result follows.

Note that without the requirement $G(u, s) \neq 0$ on ∂S, the result is false. Indeed, the harmonic functions

$$Ar^h \cos h\theta, \qquad Br^h \sin h\theta, \qquad h = 1, 2, \ldots, \qquad A, B = \text{constants},$$

satisfy the boundary condition $\partial u/\partial n = hu$ on the unit circle $r = 1$.

THEOREM 3.3. *The boundary problem*

$$Lu = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = h(s) f(u, v) \quad \text{on } \partial S, \tag{3.4}$$

where $h(s)$ is a non-negative continuous function and $f(u, v)$ is a continuously differentiable function can have no non-constant solution $[u, v]$ for which $f_u(u, v) \leqslant 0$ on ∂S.

In the identity (3.1) we set $\tau = f(u, v)$, obtaining for any solution $[u, v]$ of (3.4),

$$\int_{\partial S} h(s) f^2(u, v)\, ds = \int_S f_u(u, v)\,(Ap^2 + 2Bpq + Cq^2)\, dS,$$

from which we conclude that $h(s) f(u, v) = \partial u/\partial \nu = 0$ on ∂S since under the given assumptions the area integral is non-positive while the boundary integral is non-negative and consequently both integrals must vanish. Theorem 3.1 then yields the desired result.

Our next result shows that for a wide class of problems of the type (3.4), no solution exists. Indeed we obtain, by setting $\tau = 1/f(u, v)$ in (3.1), the identity

$$\int_{\partial S} h(s)\, ds = -\int_S \frac{f_u(u, v)}{f^2(u, v)}(Ap^2 + 2Bpq + Cq^2)\, dS,$$

which yields immediately the following:

THEOREM 3.4. *If* $\int_{\partial S} h(s)\, ds > 0$ (< 0), *then there exists no solution* $[u, v]$ *to the boundary problem* (3.4),

$$Lu = 0 \quad in\ S, \qquad \frac{\partial u}{\partial \nu} = h(s) f(u, v) \quad on\ \partial S,$$

with $f_u(u, v) \geqslant 0$ $(\leqslant 0)$ *and* $f(u, v) \neq 0$ *on* $S + \partial S$. *If* $\int_{\partial S} h(s)\, ds = 0$, *then there can only exist such a solution provided* $f_u(u, v) \equiv 0$ *in* $S + \partial S$, *i.e.*, f *is a function of* v *alone.*

The example following Corollary 3.1 shows that the preceding theorem is false without the requirement $f(u, v) \neq 0$ on $S + \partial S$.

Consider now the boundary problem

$$Lu = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = G(\varphi, s) \quad \text{on } \partial S, \tag{3.5}$$
$$\varphi = mu + nv, \qquad m, n = \text{constants},$$

studied by Dunninger [4, 5] for the special case of Laplace's equation.

If in the identity (3.1) we set $\tau = \varphi$, then for a solution $[u, v]$ of (3.5) we have the identity

$$\int_{\partial S} \varphi G(\varphi, s)\, ds = m \int_S (Ap^2 + 2Bpq + Cq^2)\, dS,$$

and arguing as in Theorem 3.3 we have

THEOREM 3.5. *If $m > 0$ and $G(\varphi, s)$ is a continuous function satisfying $\varphi G(\varphi, s) \leqslant 0$ on ∂S, then problem* (3.5) *can have only constant solutions* $[u, v]$.

A theorem for the case when $m < 0$ and $\varphi G(\varphi, s) \geqslant 0$ is obviously possible and will not be stated.

In order to discuss our next result, use of which will be made in Sec. (4, 5), we introduce the following:

DEFINITION 3.1. The function $G(\varphi, s)$ is said to have a "φ" zero if $G(c, s) = 0$ for some constant c.

THEOREM 3.6. *Let $G(\varphi, s)$ be a given continuous function, non-increasing in φ, with a "φ" zero, say $\varphi = c$. Then the boundary problem* (3.5), *with $m > 0$, can have only constant solutions* $[u, v]$.

Setting $\tau = \varphi - c$ in (3.1), we obtain

$$\int_{\partial S} (\varphi - c)\, G(\varphi, s)\, ds = m \int_S (Ap^2 + 2Bpq + Cq^2)\, dS.$$

Now, $G(\varphi, s)$ is non-increasing in φ which implies $(\varphi - c)\, G(\varphi, s) \leqslant 0$, and if $m > 0$, we find that $[u, v] =$ constant.

Suppose $f(u, v, s) = G(g(u, v), s)$ for some functions g, G and that $[u, v]$ is a solution of the boundary problem

$$Lu = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = G(g(u, v), s) \quad \text{on } \partial S. \tag{3.6}$$

Then setting $\tau = g(u, v)$ in (3.1), we obtain

$$\int_{\partial S} g(u, v)\, G(g(u, v), s)\, ds = \int_S g_u(u, v)\, (Ap^2 + 2Bpq + Cq^2)\, dS,$$

and therefore we have

THEOREM 3.7. *Suppose G is a continuous function of ξ, s and $g(u, v)$ is a continuously differentiable function of u, v. If $g_u(u, v) < 0$ and $G(\xi, s)$ satisfies $\xi G(\xi, s) \geqslant 0$ on ∂S, then the problem* (3.6) *can have only constant solutions* $[u, v]$.

To obtain our final result in this direction, we set $\tau = u$ in identity (3.1), which then yields for any solution $[u, v]$ of

$$Lu = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = f(u, v, s) \quad \text{on } \partial S, \tag{3.7}$$

the identity

$$\int_{\partial S} uf(u, v, s)\, ds = \int_S (Ap^2 + 2Bpq + Cq^2)\, dS.$$

This yields

THEOREM 3.8. *If $f(u, v, s)$ is a continuous function satisfying $uf(u, v, s) \leqslant 0$ on ∂S, then the problem* (3.7) *can have only constant solutions* $[u, v]$.

4. The Linear Equation

We now restrict ourselves to the following boundary problem

$$L_1 u = (Au_x)_x + (Bu_x)_y + (Bu_y)_x + (Cu_y)_y = 0 \quad \text{in } S, \tag{4.1}$$

$$\frac{\partial u}{\partial \nu} = G(\varphi, s) \quad \text{on } \partial S, \qquad \varphi = mu + nv, \qquad m, n = \text{constants}, \tag{4.2}$$

where the coefficients A, B, C are functions of only x and y, satisfying $B^2 - AC < 0$, $A > 0$ in S.

Upon setting $\tau = \varphi_1 - \varphi_2$, $\varphi_1 = mu_1 + nv_1$, $\varphi_2 = mu_2 + nv_2$, $f_1 = f_2 = 1$ in (2.1) we obtain

$$\begin{aligned}\int_{\partial S} (\varphi_1 - \varphi_2) \left(\frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu}\right) ds \\ = m \int_S [A(p_1 - p_2)^2 + 2B(p_1 - p_2)(q_1 - q_2) + C(q_1 - q_2)^2]\, dS,\end{aligned} \tag{4.3}$$

which forms the basis for the following:

THEOREM 4.1. *If $[u_1, v_1]$, $[u_2, v_2]$ both solve* (4.1), (4.2) *with $m > 0$, and if $G(\varphi, s)$ is a continuous function, non-increasing in φ, with no "φ" zeros, then $u_1 - u_2 = constant$.*

Indeed, from the hypotheses on the boundary conditions it is clear that the boundary integral is non-positive. In addition, the ellipticity condition and $m > 0$ imply that the right side of (4.3) is non-negative and hence must be zero which yields $p_1 - p_2 = q_1 - q_2 = 0$, or $u_1 - u_2 = \text{constant}$.

Note that without the restriction that $G(\varphi, s)$ have no "φ" zero the above result is superceded by Theorem 3.6.

It should also be remarked that the above result can be extended to allow, on part or all of the boundary, specification of φ itself instead of $\partial u/\partial \nu$; and in the case $m = 1$, $n = 0$ to allow the right side of (4.1) to be non-zero. These remarks we leave to the reader, as they are obvious extensions.

Setting $G(\varphi, s) = h(s)\,\varphi$ with $h(s) \leqslant 0$, then it follows that G has a "φ" zero, namely $c = 0$ and hence the hypotheses of Theorem 3.6 are satisfied and we obtain:

COROLLARY 4.1. *For the boundary problem*

$$L_1 u = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = h(s)\,\varphi \quad \text{on } \partial S,$$
$$\varphi = mu + nv, \qquad m, n = \text{constants},$$

where $m > 0$ *and* $h(s)$ *is a given continuous function satisfying* $h(s) \leqslant 0$ *on* ∂S, *the only possible solution is* $[u, v] = [\alpha, \beta]$ *where* α, β *are any constants such that* $m\alpha + n\beta = 0$.

If $h > 0$ then the example following Corollary 3.1 shows that non-constant solutions may exist. Moreover, it also shows that the solution need not be unique.

It is interesting to note that in the case $m = 1$, $n = 0$ a uniqueness theorem of a different nature can be obtained which is independent of the sign of h. Whether or not the following result holds for the general case is an open question.

THEOREM 4.2. (Steklov-type problem [10]). *Let* $h(s)$ *be a given continuous function on* ∂S. *If* u_2 *is a solution of the boundary problem*

$$L_1 u = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = h(s)\,u \quad \text{on } \partial S,$$

which does not vanish identically on any open subset of S, *then any other solution* u_1 *for which the ratio* $\lambda = u_1/u_2$ *is continuously differentiable in* S *and continuous in* $S + \partial S$, *is linearly dependent on* u_2.

Following Martin [9], we set $\tau = \lambda, f_1 = u_1, f_2 = u_2$ in (2.1) and obtain the integral identity

$$\begin{aligned}\int_{\partial S} \lambda \left(u_2 \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial u_2}{\partial \nu}\right) ds \\ = \int_S [A(p_1 - \lambda p_2)^2 + 2B(p_1 - \lambda p_2)(q_1 - \lambda q_2) + C(q_1 - \lambda q_2)^2]\, dS.\end{aligned} \tag{4.4}$$

The theorem is an obvious consequence of (4.4) once we note that

$$p_1 - \lambda p_2 = u_2 \lambda_x, \qquad q_1 - \lambda q_2 = u_2 \lambda_y,$$

and recall that u does not vanish identically on any open subset of S.

The example following Corollary 3.1 shows that the requirements on λ are essential.

With $G(\varphi, s) = h(s)\, e^{\varphi}$, $h(s) \leqslant 0$, it is clear that Theorem 4.1 applies. The restriction $h(s) \leqslant 0$, can be removed but we can claim uniqueness only among a restricted set of functions as is illustrated below.

THEOREM 4.3. *If* $[u_1, v_1]$ *and* $[u_2, v_2]$ *are both solutions of the boundary problem*

$$L_1 u = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = h(s)\, e^{\varphi} \quad \text{on } \partial S,$$
$$\varphi = mu + nv, \qquad m, n = \text{constants},$$

for which $(mu_1 + nv_1) - (mu_2 + nv_2) \neq 0$ *in* $S + \partial S$, *then* $u_1 - u_2 = \text{const.}$

The result follows immediately from the following identity, obtained by setting

$$\tau = \frac{1}{e^{\varphi_1} - e^{\varphi_2}}, \qquad f_1 = e^{\varphi_1}, \qquad f_2 = e^{\varphi_2},$$
$$\varphi_1 = mu_1 + nv_1, \varphi_2 = mu_2 + nv_2$$

in (2.1),

$$\int_{\partial S} \frac{e^{\varphi_2} \dfrac{\partial u_1}{\partial \nu} - e^{\varphi_1} \dfrac{\partial u_2}{\partial \nu}}{e^{\varphi_1} - e^{\varphi_2}}\, ds$$
$$= -m \int \frac{e^{\varphi_1} - e^{\varphi_2}}{(e^{\varphi_1} - e^{\varphi_2})^2} \cdot [A(p_1 - p_2)^2 + 2B(p_1 - p_2)(q_1 - q_2) + C(q_1 - q_2)^2]\, dS.$$

As our final example in this direction we set $G(\varphi, s) = h(s)\, \varphi^{p+1}$, $h(s) \leqslant 0$, where $p = 1, 2, \ldots$. Once again Theorem 3.6 applies if p is an even integer, but fails to give any information, as does Theorem 4.1, when p is an odd integer. However, by once again restricting the class of admissible functions we can remove the restrictions on both h and p as the next theorem shows.

THEOREM 4.4. *Let* $h(s)$ *be a given continuous function which does not vanish on any open subset of* ∂S. *If* $[u_1, v_1]$ *is a solution of the boundary problem*

$$L_1 u = 0 \quad \text{in } S, \qquad \frac{\partial u}{\partial \nu} = h(s)\, \varphi^{p+1} \quad \text{on } \partial S,$$
$$\varphi = mu + nv, \qquad m > 0, \qquad m, n = \text{constants}, \qquad p = 1, 2, \ldots$$

for which $\varphi_1 = mu_1 + nv_1 \neq 0$ *in* $S + \partial S$ *then no other solution* $[u_2, v_2]$ *exists for which* $\varphi_2 = mu_2 + nv_2 \neq 0$ *in* $S + \partial S$ *and* $0 < |\lambda| < 1$, *where* $\lambda = \varphi_1/\varphi_2$.

Consider the following integral identity

$$\int_{\partial S} \frac{\varphi_2^{1+p} \dfrac{\partial u_1}{\partial \nu} - \varphi_1^{1+p} \dfrac{\partial u_2}{\partial \nu}}{\varphi_2^{1+p}(1 - \lambda^p)^{1+1/p}}\, ds$$
$$= m(1 + p) \int_S \frac{\lambda^p}{\varphi_1(1 - \lambda^p)^{2+1/p}} \tag{4.5}$$
$$\cdot [A(p_1 - \lambda p_2)^2 + 2B(p_1 - \lambda p_2)(q_1 - \lambda q_2) + C(q_1 - \lambda q_2)^2]\, dS,$$

obtained by setting

$$\tau = \frac{1}{\varphi_2^{1+p}(1 - \lambda^p)^{1+1/p}}$$

in (2.1). Suppose a second solution $[u_2, v_2]$ exists. Then it is clear that the identity (4.5) exists and moreover we easily obtain

$$p_1 - \lambda p_2 = 0, \qquad q_1 - \lambda q_2 = 0 \quad \text{in} \quad S. \tag{4.6}$$

From (1.3) and the definition of λ we have

$$(\lambda_x^2 + \lambda_y^2)\varphi_2^2 = [m(p_1 - \lambda p_2) - n(B(p_1 - \lambda p_2) + C(q_1 - \lambda q_2))]^2$$
$$+ [m(q_1 - \lambda q_2) + n(A(p_1 - \lambda p_2) + B(q_1 - \lambda q_2))]^2,$$

which in view of (4.6) implies $(\lambda_x^2 + \lambda_y^2)\varphi_2^2 = 0$ and therefore $\lambda = k =$ constant. Consequently, (4.6) yields

$$u_1 = ku_2 + \ell, \qquad \ell = \text{constant in } S + \partial S. \tag{4.7}$$

This implies

$$\frac{\partial u_1}{\partial \nu} = k \frac{\partial u_2}{\partial \nu}, \qquad \varphi_1^{1+p} = k\varphi_2^{1+p} \quad \text{on } \partial S,$$

and hence

$$\lambda^{1+p} = \lambda,$$

which can hold only if $\lambda = \pm 1$ and in either case we have obtained a contradiction.

By referring to [5], it becomes evident that by making some trivial modifications, the results obtained there for the boundary problem (1.1), hold for the more general boundary problem (4.1), (4.2). In particular attention is drawn to [5, Theorem 5.1].

5. A Special Quasi-Linear Equation

Consider the following boundary problem

$$(\rho u_x)_x + (\rho u_y)_y = 0 \quad \text{in } S, \tag{5.1}$$

$$\rho \frac{\partial u}{\partial n} = G(\varphi, s) \quad \text{on } \partial S, \qquad \varphi = mu + nv, \qquad m, n = \text{constants}, \tag{5.2}$$

where

$$\rho = \rho(x, y, \| \nabla u \|), \qquad \| \nabla u \| = (u_x^2 + u_y^2)^{1/2},$$

and where we have replaced the conormal derivative by $\rho(\partial u/\partial n)$.

In general (5.1) is a nonlinear equation. However, many equations of mathematical physics which have the form (5.1) are actually only quasi-linear, since the second derivatives enter in a linear way.

Typical examples are:

I. The equation of a potential gas flow [1],

$$(\rho u_x)_x + (\rho u_y)_y = 0 \tag{5.3}$$

or

$$(c^2 - u_x^2)\, u_{xx} - 2u_x u_y u_{xy} + (c^2 - u_y^2)\, u_{yy} = 0,$$

where

$$c^2 = 1 - \frac{\gamma - 1}{2} \| \nabla u \|^2 > 0, \qquad \gamma > 1 \text{ is constant}, \qquad \rho = c^{2/\gamma - 1},$$

the equation being of elliptic type only if the flow is subsonic, i.e.,

$$\| \nabla u \|^2 < \frac{2}{\gamma + 1}.$$

Note that (5.3) is simply the equation obtained by eliminating the function v (stream function) from the first order system

$$\rho u_x = v_y\,, \qquad \rho u_y = -\, v_x\,.$$

II. The equation of minimal surfaces [2],

$$(\rho u_x)_x + (\rho u_y)_y = 0, \tag{5.4}$$

or

$$(1 + u_y^2)\, u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)\, u_{yy} = 0,$$

where

$$\rho = \frac{1}{(1 + \| \nabla u \|^2)^{1/2}}.$$

Guided by the fact that in both (5.3), (5.4) the function ρ has the properties $\rho > 0$, and $\rho \| \nabla u \|$ is non-decreasing in $\| \nabla u \|$, we are led to consider the boundary problem (5.1), (5.2) under these assumptions and in addition we require that (5.1) be quasi-linear and of elliptic type. It is immediately evident now that the results of Sec. 3 hold for the boundary problem (5.1), (5.2).

Moreover, we have the following:

THEOREM 5.1. *Let $G(\varphi, s)$ be a continuous function, $\varphi = mu + nv$, ($m > 0$, n constants), non-increasing in φ, with no "φ" zero. If $[u_1, v_1]$, $[u_2, v_2]$ are two solutions of the boundary problem* (5.1), (5.2) *where $\rho > 0$ is a given function of $x, y, \| \nabla u \|$ such that $\rho \| \nabla u \|$ is non-decreasing in $\| \nabla u \|$, then $u_1 - u_2$ is a constant.*

Setting $f_1 = f_2 = 1$, $\tau = \varphi_1 - \varphi_2$, $\varphi_1 = mu_1 + nv_1$, $\varphi_2 = mu_2 + nv_2$, $A_1 = \rho_1$, $A_2 = \rho_2$ in (2.1) we obtain the identity

$$\begin{aligned} &\int_{\partial S} (\varphi_1 - \varphi_2) \left(\rho_1 \frac{\partial u_1}{\partial n} - \rho_2 \frac{\partial u_2}{\partial n} \right) ds \\ &\qquad = m \int [\rho_1 \| \nabla u_1 \|^2 + \rho_2 \| \nabla u_2 \|^2 - (\rho_1 + \rho_2) \nabla u_1 \cdot \nabla u_2] \, dS. \end{aligned} \tag{5.5}$$

From the hypothesis on the boundary conditions the boundary integral is non-positive. In addition, by Schwarz's inequality, the monotonicity of $\rho \| \nabla u \|$ and the non-negativeness of ρ we have that

$$\begin{aligned} &\rho_1 \| \nabla u_1 \|^2 + \rho_2 \| \nabla u_2 \|^2 - (\rho_1 + \rho_2) \nabla u_1 \cdot \nabla u_2 \\ &\qquad \geqslant \rho_1 \| \nabla u_1 \|^2 + \rho_2 \| \nabla u_2 \|^2 - (\rho_1 + \rho_2) \| \nabla u_1 \| \| \nabla u_2 \| \\ &\qquad = (\rho_1 \| \nabla u_1 \| - \rho_2 \| \nabla u_2 \|)(\| \nabla u_1 \| - \| \nabla u_2 \|) \geqslant 0. \end{aligned}$$

Thus the area integral is non-negative and hence both integrals must be zero. This can only happen if

$$\| \nabla u_1 \| = \| \nabla u_2 \|, \qquad \nabla u_1 \cdot \nabla u_2 = \| \nabla u_1 \| \| \nabla u_2 \|,$$

from which it easily follows that $\nabla u_1 = \nabla u_2$ as claimed.

The proof of this theorem follows closely the proof employed by Levin [8], who proved the above result for the minimal surfaces equation under the boundary condition (5.2) with $m = 1$, $n = 0$. By referring to Levin's paper it is now evident what modifications are needed in order to generalize his results to the more general problem (5.1), (5.2), but these matters will not be pressed any further.

6. A Comparison Theorem

It is of interest to compare solutions of two different but somewhat related boundary problems. As a preliminary example in this direction we will consider two boundary problems, where the boundary conditions are independent of v. Specifically, consider the problems

$$\begin{aligned} \ell_1 u &= (au_x)_x + (bu_x)_y + (bu_y)_x + (cu_y)_y + g(x, y, u)\, u = 0 \quad \text{in } S, \\ &\qquad \frac{\partial u}{\partial \nu} + f(u, s)\, u = 0 \quad \text{on } \partial S, \end{aligned} \tag{6.1}$$

$$\begin{aligned} \ell_2 w &= (Aw_x)_x + (Bw_x)_y + (Bw_y)_x + (Cw_y)_y + G(x, y, w)\, w = 0 \quad \text{in } S, \\ &\qquad \frac{\partial w}{\partial \sigma} + F(w, s)\, w = 0 \quad \text{on } \partial S. \end{aligned} \tag{6.2}$$

Here the coefficients in (6.1) and (6.2) depend upon (x, y, u) and (x, y, w) respectively, and both satisfy the ellipticity condition. Moreover $\partial u/\partial \nu$ and $\partial w/\partial \sigma$ are the respective conormal derivatives of u and w with respect to their corresponding operators.

The basis of our result is the following formal integral identity which is a slight modification of the identity (4.4):

$$\begin{aligned} &\int_{\partial S} \lambda \left(w \frac{\partial u}{\partial \nu} - u \frac{\partial w}{\partial \sigma} \right) ds \\ &\int_S [(a - A)\, u_x^2 + 2(b - B)\, u_x u_y + (c - C)\, u_y^2 + (G - g)\, u^2]\, dS \\ &\qquad + \int_S w^2 (A\lambda_x^2 + 2B\lambda_x\lambda_y + C\lambda_y^2), \end{aligned} \tag{6.3}$$

where $\lambda = u/w$ and u and w are solutions of (6.1) and (6.2) respectively.

Theorem 6.1. *Let $f(u, s)$ and $F(w, s)$ be given continuous functions on ∂S. If w is a solution of the boundary problem* (6.2) *which does not vanish identically on any open subset of S, then every solution u of the boundary problem* (6.1) *must be a constant multiple of w provided*

(i) *$\lambda = u/w$ is continuous in S and continuously differentiable in $S + \partial S$,*

(ii) $\int_S [(a - A)\, u_x^2 + 2(b - B)\, u_x u_y + (c - C)\, u_y^2 + (G - g) u^2]\, dS \geqslant 0)$

(iii) *$F(w, s) \leqslant f(u, s)$ on ∂S.*

The proof follows immediately from the identity (6.3) and is therefore omitted.

COROLLARY 6.1. *Under the same hypothesis as above, except that condition* (ii) *is replaced by* $G \geqslant g$ *and the matrix*

$$\begin{pmatrix} a - A & b - B \\ b - B & c - C \end{pmatrix},$$

is positive semi-definite in R, the same conclusion holds.

It should be clear that similar results can be obtained for equations containing the operators in Sec. 5.

As a final observation, we note that the functions u and w of the above theorem must have the same zeros provided u is not identically zero. It is therefore suggestive that the identity (6.3) might be used to obtain comparison theorems of Sturm-type for elliptic equations. This observation will be exploited in more detail in a subsequent paper.

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