The Dirichlet-to-Neumann Map on Two-Dimensional Manifolds With Boundary

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ABSTRACT

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This paper deals with a reconstruction problem for the conformal structure of a 2-dimensional manifold with boundary assuming knowledge of the Dirichlet-to-Neumann operator. We first develop necessary results in the theory of Riemann surfaces and Banach algebras. We then use the assumed information to construct the algebra of functions continuous on the manifold and holomorphic in the interior, and then show that the manifold itself is conformally equivalent to the maximal ideal space of this algebra.

Contents

1 Preliminaries **1**

1.1 Introduction 1

1.2 General Statement 2

2 Riemann Surfaces **3**

2.1 The Dirichlet Problem 3

2.2 Other Results 9

3 Banach Algebras **12**

3.1 Basic Definitions and Results 12

3.2 The Gelfand Transform 14

4 Main Result **16**

4.1 Constructing the Desired Manifold 16

4.2 Conformal Equivalence 20

1 Preliminaries

1.1 Introduction

This paper is an exposition on one of the main results in a paper by M. I. Belishev entitled *The Calderon Problem for Two-Dimensional Manifolds by the BC Method*([Bel]), although the result was first proven by Lassas and Uhlmann [LU]. The problem assumes knowledge of the Dirichlet-to-Neumann operator Λ_g, which acts on real valued functions defined on the boundary. Using this information, we reconstruct the Banach algebra of holomorphic functions on the manifold, and then show the maximal ideal space of this Banach algebra is conformally equivalent to the manifold itself. The next two chapters of this paper deal with developing the background material needed to prove the main result, and the last chapter contains the proof itself. In Chapter 2, we discuss a few results about Riemann surfaces, including results about the Dirichlet problem and an application of the Runge Approximation Theorem. In Chapter 3, we give a brief overview of Banach algebras so that we can start talking about function algebras on manifolds. Chapter 4 then proves the main result, first by using Λ_g to construct a manifold (Ω, g) which has Λ_g as its Dirichlet-to-Neumann map, and then showing that this manifold is unique up to conformal equivalence.

1.2 General Statement

Let (Ω, g) be a smooth, compact orientable 2-dimensional Riemannian manifold with boundary Γ and metric g. Let Δ_g be the Laplace-Beltrami operator, which in local coordinates becomes

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{\det g(x)} g^{ij}(x) \frac{\partial u(x)}{\partial x^j} \right).$$

Let $\text{Harm}_g \Omega := \{u | \Delta_g u = 0 \text{ in } \text{int}\Omega\}$ be the set of harmonic functions on Ω. For $f \in C^\infty(\Gamma)$, let u^f be the solution to the Dirichlet problem with boundary value f, meaning $\Delta_g u^f = 0$ in $\text{int}\Omega$, $u^f|_\Gamma = f$; we call u^f a potential. Let $\Lambda_g : C^\infty(\Gamma) \to C^\infty(\Gamma)$ be the Dirichlet-to-Neumann map; it is defined by

$$\Lambda_g(f) = \frac{\partial u^f}{\partial \nu}|_\Gamma$$

where ν is the outward normal.

Definition 1.1. Let (Ω, g) and (Ω', g') be two smooth, compact orientable manifolds with isometric boundaries $\partial\Omega = \partial\Omega' = \Gamma$. $(\Omega, g), (\Omega', g')$ are said to be conformally equivalent if there exists a diffeomorphism $\beta : \Omega \to \Omega'$ such that $\beta(\Omega) = \Omega'$, $\beta|_\Gamma = \text{id}$, and there is a positive function $\rho \in C^\infty(\Omega)$ with $\rho|_\Gamma = 1$ such that β is an isometry of $(\Omega, \rho g)$ onto (Ω', g').

Theorem 1.2. *(Lassas and Uhlmann/Belishev) The Dirichlet-to-Neumann map determines a smooth, compact orientable 2-manifold with boundary up to conformal equivalence.*

2 Riemann Surfaces

Definition 2.1. A continuous family of maps $\Phi_x : T_x\Omega \to T_x\Omega$ is called a complex structure if

1. Φ_x is an endomorphism
2. $\langle \Phi_x(a), \Phi_x(b) \rangle = \langle a, b \rangle$ for $a, b \in T_x\Omega$
3. $\langle \Phi_x(a), a \rangle = 0$ for $a \in T_x\Omega$

A Riemann surface is a 2 dimensional manifold with a complex structure. It is a consequence of the uniformization theorem that this definition of a Riemann surface is equivalent to the definition in terms of holomorphic charts.

2.1 The Dirichlet Problem

In this section, we aim to show that the solution to the Dirichlet problem exists and is unique, which we need to know before we can start talking about the Dirichlet-to-Neumann map. Before we begin, we first state a well-known result about harmonic functions which we need.

Theorem 2.2. *(The Maximum Principle) Let u be a harmonic function on a Riemann surface X. If u attains its maximum at an interior point of X, then u is constant.*

Definition 2.3. The Dirichlet problem for a domain $X \subset \mathbb{C}$ is the following: Given a continuous function $f : \partial X \to \mathbb{R}$, we wish to find a continuous function $u : \overline{X} \to \mathbb{R}$ that satisfies:

1. $\Delta u = 0$ in X

2. $u = f$ on ∂X

We begin by first solving the Dirichlet problem for the disk $D(r) = \{z \in \mathbb{C} \big| |z| < r\}$.

Proposition 2.4. *Let $f : \partial D(r) \to \mathbb{R}$ be continuous. Then the Dirichlet problem can be solved by the Poisson integral:*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} f(re^{i\theta}) d\theta$$

for $|z| < r$, and $u(z) = f(z)$ for $|z| = r$.

This is a standard result, and a proof can be found in [For]. Since the property that a function is harmonic is preserved under a biholomorphic mapping, the Poisson integral allows us to solve the Dirichlet problem on any domain $Y \subset X$ that lies in a single holomorphic chart (U, z) and whose image $z(Y)$ is a disk in $\mathbb{C}$. More generally, if we had a domain Y whose image $z(Y)$ is a simply connected region in $\mathbb{C}$, then we can use the Riemann mapping theorem to get a holomorphic map from $z(Y)$ onto the unit disk, and we could solve the Dirichlet problem for any such $Y \subset X$. To solve the Dirichlet problem on a more general Riemann surface X, we need to introduce the notion of subharmonic functions. Let $S(X)$ denote the set of subdomains $Y \subset X$ such that the Dirichlet problem can be solved on Y for an arbitrary continuous boundary function $f : \partial Y \to \mathbb{R}$. Note that by the above remark, the set $S(X)$ is not empty. For a continuous function $u : Y \to \mathbb{R}$, let $H_Y(u)$ be the function that has boundary values $u|_{\partial Y}$ and is harmonic in Y. Note that a function u is harmonic in int$X$ exactly when $H_Y(u) = u$ for every $Y \in S(X)$. This motivates the following definition:

Definition 2.5. A continous function $u : X \to \mathbb{R}$ is called subharmonic if $H_Y(u) \geq u$ for every $Y \in S(X)$.

We call M_X the set of subharmonic functions of X and list some of its properties ([For]):

1. M_X is closed under addition and multiplication by non-negative scalars

2. M_X is closed under taking sup; if $u, v \in M_X$, then $\sup(u, v) \in M_X$

3. M_X is closed under the operator H_Y; if $u \in M_X$ and $Y \in S(X)$, then

$$\tilde{u} = \begin{cases} u \text{ on } X - Y \\ H_Y(u) \text{ on } Y \end{cases}$$

   is in M_X

We also note that there is a Maximum Principle for subharmonic functions. We now use a method developed by Perron to solve the Dirichlet problem on a more general domain following the treatment found in [For].

Lemma 2.6. *(Perron) Suppose $M \subset M_X$ is a nonempty set of subharmonic functions on X with the following properties:*

1. $u, v \in M \Rightarrow \sup(u, v) \in M$

2. $u \in M$, $Y \in S(X) \Rightarrow \tilde{u} \in M$($\tilde{u}$*defined as above*)

3. *There is a constant* $c \in \mathbb{R}$ *such that* $u \leq c$ *for every* $u \in M$

Then the function $\hat{u} : X \to \mathbb{R}$ *defined by* $\hat{u}(x) = \sup\{u(x) \big| u \in M\}$ *is harmonic in X.*

Proof. The proof requires Harnack's theorem, which states that a monotone increasing, bounded sequence of harmonic functions on a domain Y converges uniformly on compact subsets of Y to a harmonic function. Now let $x_0 \in X$ with $Y \in S(X)$ a neighborhood of x_0, and pick a sequence $u_n \in M$ such that $lim_{n\to\infty} u_n(x_0) = \hat{u_0}(x)$ and $u_0 \leq u_1 \leq u_2 \leq ....$ Then, by property 2) of M, we have

$$H_Y(u_0) \leq H_Y(u_1) \leq H_Y(u_2) \leq ...$$

is a monotone increasing, bounded sequence of harmonic functions in M, so by Harnack's theorem, the sequence converges to a harmonic $u : Y \to \mathbb{R}$ with $u(x_0) = \hat{u}(x_0)$ and $u \leq \hat{u}$. To see that $u = \hat{u}$ let $x_1 \in Y$, and similiarly let $v_n \in M$ be an increasing sequence such that $\lim_{n\to\infty} v_n(x_1) = \hat{u}(x_1)$, and we can again get a monotone increasing, bounded sequence $H_Y(v_0) \leq H_Y(v_1) \leq ...$ converging to some harmonic v. Because of property 1) of M, we may assume this new sequence also satisfies $H_Y(u_n) \leq H_Y(v_n)$. Therefore the harmonic function v satisfies $u \leq v \leq \hat{u}$, so then, in particular, $u(x_0) = v(x_0) = \hat{u}(x_0)$. If we then apply the Maximum Principle to the harmonic function $v - u$, we have $v = u$ everywhere on Y, so then

$u(x_1) = v(x_1) = \hat{u}(x_1)$ on Y. Since $x_1 \in Y$ was arbitrary, we have $u(x) = \hat{u}(x)$ for all $x \in Y$, and $\hat{u}$ is harmonic in Y. □

Definition 2.7. Let $f : \partial X \to \mathbb{R}$ be a continuous bounded function. Let B_f be the set of continuous functions $u : X \to \mathbb{R}$ such that:

1. $u|_{\text{int}X}$ is subharmonic
2. $u|_{\partial X} \leq f$

We call B_f the Perron class of f.

The set B_f satisfies the conditions of 2.6, with the constant $c = \sup\{f(x)\big| x \in \partial X\}$, so we know $\hat{u} = \sup\{u \in B_f\}$ is harmonic in int$X$. The question left is whether or not $\hat{u}$ has the desired boundary behavior, meaning does $\hat{u}$ satisfy

$$\lim_{y \to x} \hat{u}(y) = f(x), y \in \text{int}X, x \in \partial X?$$

Definition 2.8. A boundary point $x \in \partial X$ is called regular if there is an open neighborhood U of x and a continuous function $\beta : X \cap U \to \mathbb{R}$ such that:

1. $\beta|_{\text{int}X \cap U}$ is subharmonic.
2. $\beta(x) = 0$, and $\beta(y) < 0$ for all $y \in X \cap U / \{x\}$.

We call β a barrier at x.

Lemma 2.9. *Let $x \in \partial X$ be a regular boundary point, V a neighborhood of x, and $m, c \in \mathbb{R}$ with $m \leq c$. Then there exists a function $v \in C(X)$ such that $v|_{\text{int}X}$ is subharmonic, $v(x) = c$, $v|_{X \bigcap V} \leq c$, and $v|_{X - V} = m$.*

Proof. Without loss of generality, let $c = 0$. Let U be a neighborhood of x with barrier $\beta \in C(X \bigcap U)$. We may also assume $V \subset U$, by shrinking V if necessary. Then we have that $\sup\{\beta(x) \big| x \in \partial V \bigcap X\} < 0$, so there is a constant $k > 0$ such that $k\beta|_{\partial V \bigcap} < m$. Then define

$$v = \begin{cases} \sup(m, k\beta) \text{ on } X \bigcap V \\ \\ m \text{ on } X - V \end{cases}$$

and this v satisfies the conditions of the lemma by construction. □

Lemma 2.10. *Let $f : \partial X \to \mathbb{R}$ be a bounded continuous function and let $\hat{u} = \sup\{u \in B_f)\}$, where B_f is the Perron class of f. Then for every regular boundary point $y \in \partial X$, we have*

$$\lim_{y \to x} \hat{u}(y) = f(x) \ y \in \text{int}X, x \in \partial X.$$

Proof. Fix $\epsilon > 0$, and let V be a relatively compact open neighborhood of $x \in \partial X$ such that $f(x) - \epsilon \leq f(y) \leq f(x) + \epsilon$ for all $y \in \partial X \bigcap V$. Let k_1, k_2 be constants such that $k_1 \leq f(y) \leq k_2$ for all $y \in \partial X$. By 2.9, pick a function $v \in C(X)$ which is subharmonic on int$X$ that satisfies $v|_{\partial X} = f - \epsilon$, $v|_{X \bigcap V} \leq f - \epsilon$, $v|_{X-V} = k - \epsilon$. Then $v \in B_f$, so we have

$$\liminf_{y \to x} \hat{u}(y) \geq f(x) - \epsilon \ y \in \text{int}X, x \in \partial X.$$

Again by 2.9, we have a function $w \in C(X)$ which is subharmonic on int$X$ and satisfies $w|_{\partial X} = -f$, $w|_{X \bigcap V} \leq -f$, and $w|_{X-V} = -k_2$. Let $u \in B_f$; since $u(y) \leq f(x) + \epsilon$ for all $y \in \partial X \bigcap V$, we have $u(y)+w(y) \leq \epsilon$ for $y \in \partial X \bigcap V$, and $u(y)-w(y) \leq k_2-k_2 = 0$ for $y \in X \bigcap \partial V$. Then by the Maximum Principle, $u+w \leq \epsilon$ on $X \bigcap V$. Since $u \in B_f$ is arbitrary, we then have

$$\limsup_{y \to x} \hat{u}(y) \leq \epsilon - w(x) = f(x) + \epsilon \ y \in \text{int}X, x \in \partial X$$

This proves the lemma. □

Theorem 2.11. *Let X be Riemann surface with regular boundary. Then for every bounded, continuous function $f : \partial X \to \mathbb{R}$, the Dirichlet problem can be solved.*

Proof. This follows directly from 2.6 and 2.10. □

Lemma 2.12. *The solution to the Dirichlet problem is unique.*

Proof. Let X be a Riemann surface and $f : \partial X \to \mathbb{R}$ be continuous. Suppose u_1, u_2 are solutions to the Dirichlet problem for f. Then $u_1 - u_2 = 0$ on ∂X, so by the Maximum Principle for harmonic functions, $u_1 = u_2$. □

2.2 Other Results

Theorem 2.13. *Let Ω be a compact Riemann surface with smooth boundary Γ, and ω a one-form on Ω. Then*

$$\int_\Omega d\omega = \int_\Gamma \omega.$$

We will need the following criterion for holomorphy; recall the differential operator $\bar{\partial} f = \frac{1}{2}(f_x + i f_y)$ for (analytic) coordinates x, y.

Theorem 2.14. *Let $U \subset X$ be an open subset of a Riemann surface X. If $\bar{\partial} f = 0$ on U, then f is holomorphic on U.*

We lastly need to use an application of the Runge Approximation Theorem.

Theorem 2.15. *(The Runge Approximation Theorem) Suppose X is a non-compact Riemann surface and $Y \subset X$ is an open subset whose complement contains no relatively compact connected component. Then every holomorphic function on Y can be approximated uniformly on every compact subset of Y by holomorphic functions on X.*

For a Riemann surface X, let $x \in X$, U be an open neighborhood x, and f be a function holomorphic on U. We define an equivalence relation on such pairs (U, f) by saying $(U, f) \sim (V, g)$ if there exists a neighborhood $W \subset U \bigcap V$ of x such that $f|_W = g|_W$.

Definition 2.16. An equivalence class, denoted f_x, of the above equivalence relation is called the germ of a holomorphic function at x. We call $\mathcal{O}_x$ the ring of all germs at x.

Theorem 2.17. *Let X be a non-compact Riemann surface. Let $\mathcal{O}(X)$ be the algebra of holomorphic functions on X with $f_1, ..., f_n \in \mathcal{O}(X)$. Then $g \in \mathcal{O}(X)$ belongs to the ideal $(f_1, ..., f_n)$ if and only if the germ g_x belongs to the ideal $(f_{1,x}, ..., f_{n,x})$ for all $x \in X$.*

A proof of this can be found in [Nar]. The result that we are after is the following:

Lemma 2.18. *Let X be a compact Riemann surface with boundary ∂X, and $\mathcal{A}(X)$ be the algebra of functions holomorphic in the interior of X and continuous on ∂X. Let $I \subset \mathcal{A}(X)$ be a closed ideal. If $I = (f_1, ..., f_n)$ is finitely generated and f_j have no common zeroes in X, then $I = \mathcal{A}(X)$.*

Proof. We will show that there exist functions $\hat{g}_j \in \mathcal{A}(X), \hat{f}_j \in I$ such that

$$\sum_{j=1}^{n} \hat{g}_j \hat{f}_j = 1.$$

If one tries to use the given functions f_j that generate I to produce the constant function 1, a difficulty arises in the fact that the f_js are only continuous on the boundary ∂X. To get around this, we embed X into a slightly larger noncompact Riemann surface $\tilde{X}$. By the Runge Approximation theorem, we can approximate the functions $\{f_j\}$ by functions $\{\tilde{f}_j\}$ in $\mathcal{O}(\tilde{X})$. Furthermore, we can pick these approximating functions $\{\tilde{f}_j\}$ so that they still have no common zeroes; since I is closed, we can also ensure that our approximating functions satisfy $\{\tilde{f}_j\}|_X \in I$. Now pick $x \in \tilde{X}$, and let $\tilde{f}_j(x) \neq 0$. Then the constant function is in the germ $\tilde{f_{j,x}}$, and thus the ideal $(\tilde{f_{j,x}})$ is actually the entire ring $\mathcal{O}_x$. Since this holds for all $x \in X$, by 2.17, we have that $1 \in (\tilde{f}_1, ..., \tilde{f}_n)$, so there exists $\tilde{g}_j \in \mathcal{O}(\tilde{X})$ such that

$$\sum_{j=1}^{n} \tilde{g}_j \tilde{f}_j = 1$$

Let $\tilde{g}_j|_X = \hat{g}_j, \tilde{f}_j|_X = \hat{f}_j$, and we have $1 \in I$. □

3 Banach Algebras

3.1 Basic Definitions and Results

We begin with a few basic definitions and results about Banach algebras that we will later need. We recall that a (unital) Banach algebra is a normed algebra $\mathcal{A}$ with unit $\mathbf{1}$ (and $\|\mathbf{1}\| = 1$) that is complete with respect to the norm and satisfies $\|xy\| \leq \|x\|\|y\|$ for all x, $y \in \mathcal{A}$. For our purposes, we assume the algebras are associative and over the field $\mathbb{C}$. A Banach algebra is called commutative if $xy = yx$ for all x,y in $\mathcal{A}$. An important example of a commutative Banach algebra is $C(X)$= continuous functions on a compact Hausdorff space X with norm $\|f\| = \max\{|f(x)|\big|x \in X\}$, and addition/multiplication defined pointwise. We call subalgebras of $C(X)$ function algebras.

Definition 3.1. An isomorphism $\Psi : A(X) \to B(Y)$ of function algebras is called spatial if there exists a bijection $\alpha : X \to Y$ that satisfies $\Psi(w) = w \circ \alpha^{-1}$ for all $w \in A(X)$.

Definition 3.2. Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. We call the set $\sigma(x) = \{\lambda \in \mathbb{C}\big|\lambda\mathbf{1} - x$ is not invertible in $\mathcal{A}\}$ the spectrum of x. We call $r(x) = \sup\{|\lambda|\big|\lambda \in \sigma(x)\}$ the spectral radius of x.

We mention the general fact that $\sigma(x) \neq \emptyset$ for $x \in \mathcal{A}$.

Theorem 3.3. *(Gelfand-Mazur) If $\mathcal{A}$ is a Banach algebra that is also a division algebra, then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $\lambda \in \sigma(x)$. Since $\lambda\mathbf{1} - x \in \mathcal{A}$ is not invertible, it must be zero, so $\lambda\mathbf{1} = x$. Then the map $\psi : \mathbb{C} \to \mathcal{A}$ given by $\psi(\lambda) = \lambda\mathbf{1}$ is linear, bijective, and preserves multiplication and norm. □

Definition 3.4. A linear functional ϕ on an algebra $\mathcal{A}$ is called multiplicative if it is nontrivial and $\phi(xy) = \phi(x)\phi(y)$ for all x and y in $\mathcal{A}$

Proposition 3.5. *If $\mathcal{A}$ is a commutative Banach algebra, then there is a one-to-one correspondence between the set of multiplicative linear functionals on $\mathcal{A}$ and the set of maximal ideals of $\mathcal{A}$ by $\phi \leftrightarrow \ker\phi$.*

Proof. First, let ϕ be a multiplicative linear functional on $\mathcal{A}$, and consider $\ker\phi$. Since ϕ is nontrivial by definition, $\ker\phi$ is a proper ideal. To show it's maximal, let $x \in \mathcal{A} - \ker\phi$, and observe

$$\phi\left(\mathbf{1} - \frac{x}{\phi(x)}\right) = 1 - \phi\left(\frac{x}{\phi(x)}\right) = 1 - \frac{1}{\phi(x)}\phi(x) = 0.$$

Then we have

$$\mathbf{1} = \left(\mathbf{1} - \frac{x}{\phi(x)}\right) + \frac{x}{\phi(x)}.$$

with the element in parenthesis being in $\ker\phi$ and $\frac{x}{\phi(x)} \notin \ker\phi$. Thus adjoining x to $\ker\phi$ generates 1, and hence the entire algebra $\mathcal{A}$, so $\ker\phi$ is a maximal ideal. Now suppose I is a maximal ideal in $\mathcal{A}$; then $\mathcal{A}/I$ is a division algebra, and by the Gelfand-Mazur Theorem, there exists $\psi : \mathcal{A}/I \to \mathbb{C}$ that is an isometric isomorphism. If $\pi : \mathcal{A} \to \mathcal{A}/I$ is the projection map, then $\psi \circ \pi$ is a multiplicative linear functional with kernel I. To see the correspondence is 1-1, let ϕ_1, ϕ_2 be multiplicative linear functionals with kerϕ_1 = kerϕ_2. Let $x \in \mathcal{A}$, and then

$$(\phi_1(x) - \phi_2(x))\,\mathbf{1} = (x - \phi_2(x)\mathbf{1}) - (x - \phi_1(x)\mathbf{1})$$

The first term on the right is in ker$\phi_2 = I$, and the second term is in ker$\phi_1 = I$. Then the term on the left is in I, so $\phi_1(x) = \phi_2(x)$, and $\phi_1 = \phi_2$. □

We will denote the set of maximal ideals of a Banach algebra $\mathcal{A}$ by $\mathcal{M}_{\mathcal{A}}$.

3.2 The Gelfand Transform

Definition 3.6. Let $\mathcal{A}$ be a Banach algebra. The Gelfand transform is the mapping $G : \mathcal{A} \to C(\mathcal{M}_{\mathcal{A}})$

$$G(x)(\phi) = \phi(x)$$

for $x \in \mathcal{A}, \phi \in \mathcal{M}_{\mathcal{A}}$. We will denote the Gelfand transform of an element x by x_G.

Definition 3.7. The Gelfand topology is the weakest topology on $\mathcal{M}_{\mathcal{A}}$ such that every $G(x)$ is continuous.

In general, the Gelfand transform is a contractive homomorphism; the norm we use on $C(\mathcal{M}_{\mathcal{A}})$ is $\|x_G\| = \sup\{|\phi(x)| \big| \phi \in \mathcal{M}_{\mathcal{A}}\}$. It can be shown that $x \in \mathcal{A}$ is invertible if and only if x_G is invertible in $C(\mathcal{M}_{\mathcal{A}})$ ([Zhu]).

Proposition 3.8. *The Gelfand transform is an isometric isomorphism of $\mathcal{A}$ onto its image if $\|x^2\| = \|x\|^2$ for all $x \in \mathcal{A}$.*

Proof. To prove this, we need the spectral radius formula, which states that $r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ [Zhu]. Now, since $\lambda\mathbf{1} - x$ is not invertible in $\mathcal{A} \Leftrightarrow \lambda\mathbf{1} - x_G$ is not invertible in $C(\mathcal{M}_\mathcal{A})$, we have

$$\sigma(x) = \sigma(x_G) = \text{Range}(x_G).$$

To justify the last equality, note that $\lambda \in \text{Range}(x_G) \Leftrightarrow \phi(x) = \lambda$ for some $\phi \in \mathcal{M}_\mathcal{A} \Leftrightarrow \phi(\lambda\mathbf{1}) - \phi(x) = 0 \Leftrightarrow \phi(\lambda\mathbf{1} - x_G) = 0 \Leftrightarrow \lambda\mathbf{1} - x_G$ is not invertible in $C(\mathcal{M}_\mathcal{A}) \Leftrightarrow \lambda \in \sigma(x_G)$. From the equality $\sigma(x) = \text{Range}(x_G)$ we get the result $r(x) = \|x_G\|$. Then since $\|x^2\| = \|x\|^2$, by induction we have $\|x^{2^n}\| = \|x\|^{2^n}$. By the spectral radius formula, we have

$$\|x_G\| = r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|x\|^{2^n \frac{1}{2^n}} = \|x\|.$$

Therefore the Gelfand transform is norm preserving, and therefore an isometric isomorphism onto its image. □

Corollary 3.9. *The Gelfand transform of a function algebra on a compact space is an isometric isomorphism.*

Proof. For w in a function algebra $C(X)$, let $x \in X$ be the point that maximizes $|w(x)|$. Then we have

$$\|w\|^2 = |w(x)|^2 = |w(x)^2| = \|w^2\|.$$

The corollary is then true by the preceding proposition. □

4 Main Result

Let us restate the main theorem:

Theorem 4.1. *The Dirichlet-to-Neumann map determines a smooth, compact orientable 2-manifold up to conformal equivalence.*

4.1 Constructing the Desired Manifold

Our approach will be to use Λ_g to construct a manifold (Ω, g) with boundary Γ which has Λ_g as its Dirichlet-to-Neumann map. We then show that any other manifold (Ω', g') with the same boundary Γ, which also has Λ_g as its Dirichlet-to-Neumann map, is conformally equivalent to the constructed manifold (Ω, g). For simplicity, we consider only the case where Γ is homeomorphic to S^1. We begin by introducing coordinates γ, ν near the boundary Γ, where γ is the natural parameter on Γ, and ν the outward unit normal. In these coordinates, the Cauchy-Riemann equations for a holomorphic function $f = u + iv$ are then $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \gamma}, \frac{\partial u}{\partial \gamma} = -\frac{\partial v}{\partial \nu}$. Let Jf denote the primitive function of f with mean zero, i.e. $\frac{\partial}{\partial \gamma} Jf = f, \int_\Gamma Jf(\gamma) d\gamma = 0$. Let $\mathcal{A}(\Omega)$ be the algebra of functions holomorphic in intΩ and continuous on the boundary Γ. We first show that the Dirichlet-to-Neumann map Λ_g determines the trace algebra $\mathcal{A}(\Gamma) := \mathcal{A}(\Omega)|_\Gamma$.

Lemma 4.2. $f + (J\Lambda)^2 f = 0$ *if and only if* f *is the real part of the boundary value of a holomorphic function on* Ω.

Proof. First, suppose there exists some g such that $f+ig$ are the boundary values of a holomorphic function $u+iv$ on Ω. Then, by one of the Cauchy-Riemann equations, we have

$$\Lambda_g f = \frac{\partial u}{\partial \nu}|_\Gamma = \frac{\partial v}{\partial \gamma}|_\Gamma = \frac{\partial g}{\partial \gamma}.$$

Taking primitives of $\Lambda_g f = \frac{\partial g}{\partial \tau}$ then gives $g = J\Lambda_g f$. By the other Cauchy-Riemann equation,we have

$$\frac{\partial g}{\partial \nu} = \frac{\partial v}{\partial \nu}|_\Gamma = -\frac{\partial u}{\partial \gamma}|_\Gamma = -\frac{\partial f}{\partial \gamma}.$$

and again taking primitives gives $J\Lambda_g g = -f$. Thus we have

$$f = -J\Lambda_g g = -J(\Lambda_g \circ J\Lambda_g f) = -(J\Lambda_g)^2 f.$$

For the other direction, suppose $f = -(J\Lambda_g)^2 f$. We then solve the Dirichlet problem to get functions u, v harmonic in intΩ with boundary values $u|_\Gamma = f$, $v|_\Gamma = J\Lambda_g f$, and let $w = u+iv$. Then, along Γ, we have

$$\frac{\partial u}{\partial \nu} = \Lambda_g f = \frac{\partial v}{\partial \gamma},$$

$$\frac{\partial v}{\partial \nu} = \Lambda_g(J\Lambda_g f) = \frac{\partial}{\partial \gamma} J\Lambda_g \circ J\Lambda_g f = -\frac{\partial f}{\partial \gamma} = -\frac{\partial u}{\partial \gamma}.$$

Therefore, along the boundary Γ, the Cauchy-Riemann equations hold for w. To see that w is holomorphic in the interior of Ω, we will use the fact that $\bar{\partial} w = 0$ implies w is holomorphic, and $d\omega \wedge \omega' = d(\omega\omega') - \omega \wedge d\omega'$ for differential forms ω, ω', where

here $d = \partial + \bar{\partial}$.

$$\begin{aligned}\int_{\Omega} \left\|\bar{\partial}(u+iv)\right\|^2 &= \int_{\Omega} \bar{\partial}(u+iv) \wedge^{*} \bar{\partial}(u+iv) \\ &= \int_{\Omega} d(u+iv)^{*}\bar{\partial}(u+iv) - \int_{\Omega} (u+iv)\bar{\partial}^{*}\bar{\partial}(u+iv) \\ &= \int_{\Gamma} (u+iv)^{*}\bar{\partial}(u+iv) - \int_{\Omega} (u+iv)^{*}\Delta(u+iv) \\ &= 0\end{aligned}$$

The first integral is 0 since $\bar{\partial}(u+iv) = 0$ along Γ. The second integral is 0 since $*\Delta(u+iv) = 0$. Then $\int_{\Omega} \left\|\bar{\partial}(w)\right\|^2 = 0$, and we have $\bar{\partial}(w) = 0$ in Ω. Thus $w = u+iv$ is a holomorphic function on Ω whose real part has boundary values f. □

Lemma 4.3. *$\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Omega)$ are isometrically isomorphic as algebras.*

Proof. We first observe that a holomorphic function w on Ω cannot achieve its maximum at an interior point. Suppose $p \in \mathrm{int}\Omega$ was such that $|w(p)|$ was maximal. Let U be an open neighborhood of p; since holomorphic mappings are open mappings, then $w(U)$ is an open set in $\mathbb{C}$, contradicting $|w(p)|$ as a maximum. Therefore $\|w\|_{\mathcal{A}(\Omega)} = \|\mathrm{tr}w\|_{\mathcal{A}(\Gamma)}$, where $\mathrm{tr}w$ denotes $w|_{\Gamma}$. The mapping $\mathrm{tr} : \mathcal{A}(\Omega) \to \mathcal{A}(\Gamma)$ is then a norm preserving isomorphism. □

Lemma 4.4. *$\mathcal{M}_{\mathcal{A}(\Omega)}$ is homeomorphic to Ω.*

Proof. Consider the map $\alpha : \Omega \to \mathcal{M}_{\mathcal{A}(\Omega)}$, defined by $\alpha(p) = M_p$, where $M_p = \{f \in \mathcal{A}(\Omega) \big| f(p) = 0\}$. We know that M_p is a maximal ideal because it is the kernel of the multiplicative linear functional ϕ_p = point evaluation at p, and so by the one-to-one correspondence between maximal ideals of $\mathcal{A}(\Omega)$ and the multiplicative linear functionals on Ω discussed in 3.5, we know M_p is a maximal ideal. Showing α is surjective amounts to showing that every maximal ideal of $\mathcal{A}(\Omega)$ is of the form M_p. Now suppose there was a maximal ideal M that was not of the form M_p. Then $\bigcap_{f \in M} z(f) = \emptyset$, where $z(f) = \{p \in \Omega \big| f(p) = 0\}$. But then $\{z(f)^C\}_{f \in M}$ cover Ω, so by compactness, only finitely many are needed to cover Ω, and we have $\bigcap_{j=1}^{m} z(f_j) = \emptyset$. Now, by 2.18, we now have that $1 \in M$, contradicting M being a maximal ideal. Therefore surjectivity of α is established. To see α is injective, we make use of the identification $M_p \leftrightarrow \phi_p$ (recall $\phi_p : \mathcal{A}(\Omega) \to \mathbb{C}$ is point evaluation at p). If $\alpha(p) = \alpha(q)$, then $\phi_p = \phi_q$, meaning $f(p) = f(q)$ for every $f \in \mathcal{A}(\Omega)$, so $p = q$. $\mathcal{M}_{\mathcal{A}(\Omega)}$ with the Gelfand topology immediately gives us continuity of α. □

Lemma 4.5. *The Gelfand transform of* $\mathcal{A}(\Omega)$ *and* $\mathcal{A}(\Gamma)$ *are both spatially isomorphic to* $\mathcal{A}(\Omega)$.

Proof. We'll first show the statement is true for $\mathcal{A}(\Omega)$; recall the Gelfand transform of an element w is denoted w_G. By above we have the bijection $\alpha : \Omega \to \mathcal{M}_{\mathcal{A}(\Omega)}$. We note that by 3.5 and the characterization $\mathcal{M}_{\mathcal{A}(\Omega)} = \{M_p \big| p \in \Omega\}$ found above that every element in $\mathcal{M}_{\mathcal{A}(\Omega)}$ is of the form ϕ_p. The Gelfand transform of $w \in \mathcal{A}(\Omega)$ is then $w_G(\phi_p) = \phi_p(w) = w(p) = w(\alpha^{-1}(M_p))$ (again making use of the identification $\phi_p \leftrightarrow M_p$), so then

$$w_G = w \circ \alpha^{-1}.$$

Since the Gelfand transform is an isomorphism (3.9) and α is a bijection, the Gelfand transform of $\mathcal{A}(\Omega)$ is a spatial isomorphism. The case for $\mathcal{A}(\Gamma)$ is then true by the fact that $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Gamma)$ are isometrically isomorphic. The spatial isomorphism is $G \circ \mathrm{tr} : \mathcal{A}(\Omega) \to G(\mathcal{A}(\Gamma))$ with the bijection $\delta : \Omega \to \mathcal{M}_{\mathcal{A}(\Gamma)}$ given by $\delta(p) = \mathrm{tr}(M_p)$. □

4.2 Conformal Equivalence

So we've now constructed our manifold Ω which has Λ_g as its Dirichlet-to-Neumann map, but we still have yet to put a metric g on Ω.

Definition 4.6. A metric g conforms to the complex structure determined by the algebra $\mathcal{A}(\Omega)$ if $\mathrm{Re}\mathcal{A}(\Omega) \subset \mathrm{Harm}_g(\Omega)$.

We already have a complex structure on Ω from $\mathcal{A}(\Omega)$, and we can construct g to conform to our given structure by the following:

Definition 4.7. Let Ω be a Riemannian manifold and $\{U_k\}$ an open cover Ω. A partition of unity subordinate to $\{U_k\}$ is a family of differentiable functions $\eta_k : \Omega \to \mathbb{R}$ that satisfy:

1. $0 \leq \eta_k \leq 1$ for every k
2. $\mathrm{Supp}(\eta_k) \subset U_k$
3. For every $p \in \Omega$, there is a neighborhood V such that $V \bigcap \mathrm{Supp}(\eta_k) \neq \emptyset$ for only finitely many k.

4. $\sum_k \eta_k = 1$

Theorem 4.8. *Let Ω be a Riemannian manifold and $\{U_k\}$ an open cover of Ω. Then there exists a partition of unity subordinate to $\{U_k\}$.*

Let $\{U_k, \phi_k\}$ be an atlas on Ω with $\phi_k : U_k \to \mathbb{R}^2$ defined by $(\text{Re}w_k(x), \text{Im}w_k(x))$, with $w_k \in \mathcal{A}(\Omega)$. Let η_k be a partition of unity, so $\eta_k \geq 0, \text{Supp}\eta_k \subset U_k$, and $\sum_k \eta_k = 1$. Let $g_{ij}^{(k)} = \eta_k \delta_{ij}$ (where δ_{ij} is the Kronecker delta function), and then the tensor $g = \sum_k g^{(k)}$ is a metric conforming to the structure determined by $\mathcal{A}(\Omega)$. The algebra $\mathcal{A}(\Omega)$ is immune to conformal deformation of g by $g' = \rho g$ since, in two dimensions, $\Delta_g = 0$ and $\Delta_{\rho g} = 0$ are equivalent, so both metrics determine the same set of harmonic functions. We note the fact that two metrics conforming to the same complex structure differ by a functional multiplier. We're now ready for the last step.

Lemma 4.9. *Let (Ω, g) be the manifold just constructed which has Dirichlet-to-Neumann map Λ_g and boundary Γ. If (Ω', g') is another smooth, compact orientable 2-manifold with the same boundary Γ and the same Dirichlet-to-Neumann map Λ_g, then (Ω, g) and (Ω', g') are conformally equivalent.*

Proof. First, since the algebra $\mathcal{A}(\Gamma)$ is determined by the Dirichlet-to-Neumann map (4.2), we know that if (Ω, g) and (Ω', g') have the same Dirichlet-to-Neumann map Λ_g, then we have the equality $\mathcal{A}(\Gamma) = \mathcal{A}'(\Gamma)$. By 4.5, we have, since $\mathcal{A}(\Gamma)$ and $\mathcal{A}'(\Gamma)$ have the same Gelfand transform:

$$\mathcal{A}(\Omega) = G(\mathcal{A}(\Gamma)) = G(\mathcal{A}'(\Gamma)) = \mathcal{A}(\Omega'),$$

where all of the equalities above are spatial isomorphisms. The spatial isomorphism between $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Omega')$ gives us the bijection $\beta = \alpha'^{-1} \circ \epsilon \circ \alpha : \Omega \to \Omega'$, where α, α' are the homeomorphisms $\alpha : \Omega \to \mathcal{M}_{\mathcal{A}(\Gamma)}$, $\alpha' : \Omega' \to \mathcal{M}_{\mathcal{A}'(\Gamma)}$ we get from 4.4, and $\epsilon : \mathcal{M}_{\mathcal{A}(\Gamma)} \to \mathcal{M}_{\mathcal{A}'(\Gamma)}$ is the bijection induced by $\mathcal{A}(\Gamma) = \mathcal{A}'(\Gamma)$. Since the differentiable structures on Ω, Ω' are determined by the algebras $\mathcal{A}(\Omega), \mathcal{A}(\Omega')$ which are spatially isomorphic, we have that β is a diffeomorphism. For $p \in \Gamma$, we have $\alpha(p) = M_p = M'_p = \alpha'(p)$, so $\beta(p) = \alpha^{-1}(\alpha'(p)) = p$, and therefore $\beta|_\Gamma = id$. Now introduce a metric $\hat{g}$ on Ω so that β is an isometry. This metric conforms to the same structure determined by $\mathcal{A}(\Omega)$, so $\hat{g} = \rho g$, for some $\rho \in C^\infty(\Gamma)$. Since β is an isometry (with respect to the metric $\hat{g}$), we have $\hat{g}|_\Gamma = g'|_\Gamma = g|_\Gamma$, so $\rho|_\Gamma = 1$. Therefore $(\Omega, g), (\Omega', g')$ are conformally equivalent. $\square$

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