Math 111, Trigonometric Equation Problems

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Here are some additional problems where you solve trigonometric equations. For each of the below equations, find all x that make the statement true. Refer to the online notes for some worked examples.

1. $\sin x \tan x = \sin x$, $x \in [0, 2\pi]$

   Solution: Subtracting $\sin x$ from both sides, we have

   $$\begin{aligned} \sin(x)\tan x - \sin x &= 0 \\ \sin x(\tan x - 1) &= 0 \end{aligned}$$

   $$\Rightarrow \sin x = 0 \qquad \text{or} \qquad \tan x - 1 = 0$$

   In $[0, 2\pi]$, $\sin x = 0$ for $x = 0, \pi, 2\pi$, and $\tan x = 1$ for $x = \pi/4, 5\pi/4$ (remember $\tan x$ is positive for x terminating in quadrants I and III). Then our solutions are

   $$x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$$

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2. $\tan^2 x + \tan x - 2 = 0$

   Solution: We recognize this as a quadratic in the variable $\tan x$. Then using the quadratic formula, we get the roots of the above equation are

   $$\tan x = \frac{-1 \pm \sqrt{1 - 4(-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}$$

   Therefore the roots of the quadratic equation are $\tan x = -2, \tan x = 1$. These are the equations we now need to solve, but we'll first write the quadratic in its factored form to see why we are solving them.

   $$\tan^2 x + \tan x - 2 = (\tan x - 1)(\tan x + 2) = 0$$

   Since $\tan x$ is giving us our solutions, we will find all solutions in $(-\pi/2, \pi/2)$ (there will be one from each equation) and add integer multiples of π, the period of tangent. $\tan x = 1$ is true for $x = \pi/4$, and $\tan x = -2$, taking arctan of both sides gives us $x = \arctan(-2)$. Note in this example we only have to look for one solution to each equation since $\tan x$ is one-to-one on one cycle, here $(-\pi/2, \pi/2)$. Then we add πn to each solution to find all solutions:

   $$x = \begin{cases} \frac{\pi}{4} + \pi n \\ \arctan(-2) + \pi n \end{cases}$$

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3. $\cos(2x) = \cos x$

Solution: We want to use a double angle identity for $\cos 2x$ that will leave us with factors of $\cos x$; so we use $\cos 2x = 2\cos^2 x - 1$. Substituting this in, we have:

$$\begin{aligned} 2\cos^2 x - 1 &= \cos x \\ 2\cos^2 x - \cos x - 1 &= 0 \end{aligned}$$

We now factor using the quadratic formula; we have

$$\cos x = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)} = \frac{1 \pm 3}{4}$$

Therefore the solutions to the *quadratic* equation (note these are *not* solutions to our problem) are $\cos x = 1, -1/2$:

$$2\cos^2 x - \cos x - 1 = (\cos x - 1)\left(\cos x + \frac{1}{2}\right) = 0$$

So we first find the solutions to $\cos x = 1$ in $[0, 2\pi)$, which is only $x = 0$ (since this is the maximum of $\cos x$. The solutions to $\cos x = -1/2$ in $[0, 2\pi)$ are $x = 2\pi/3, 4\pi/3$. Then adding $2\pi n$ to all our solutions, we have

$$x = \begin{cases} 0 + 2\pi n \\ \frac{2\pi}{3} + 2\pi n \\ \frac{4\pi}{3} + 2\pi n \end{cases}$$

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4. $4 \sin x \cos x = \sqrt{3}$

Solution: We recognize the identity $\sin(2x) = 2\sin x \cos x$, and use this to rewrite the expression in terms of $\sin(2x)$. We first divide each side by 2:

$$\begin{aligned} 2\sin x \cos x &= \frac{\sqrt{3}}{2} \\ \sin(2x) &= \frac{\sqrt{3}}{2} \end{aligned}$$

Now we know how to find the solutions to $\sin\theta = \sqrt{3}/2$; the solutions in $[0, 2\pi)$ are $\theta = \pi/3, 2\pi/3$. So letting $\theta = 2x$, we get $x = \pi/6, \pi/3$. Now here is where we would normally add $2\pi n$ to each solution; however here the function that is giving us our solutions is $\sin(2x)$, not $\sin x$, and the period of $\sin(2x)$ is π. So we actually need to add integer multiples of π instead:

$$x = \begin{cases} \frac{\pi}{6} + \pi n \\ \frac{\pi}{3} + \pi n \end{cases}$$

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5. $\cos 2x = 2(\sin x - 1)$

Solution: This one is actually more difficult than I had imagined. As we'll see the numbers we get in the end aren't "nice". We'll start by rewriting $\cos(2x)$ by a double angle identity. We have three versions to choose one, and the one we want is the one that involves only factors of $\sin x$ (since those factors are already present in our equation).

$$\begin{aligned} \cos 2x &= 2(\sin x - 1) \\ 1 - 2\sin^2 x &= 2\sin x - 2 \\ -2\sin^2 x - 2\sin x + 3 &= 0 \end{aligned}$$

Now we use the quadratic formula to factor this:

$$\sin x = \frac{2 \pm \sqrt{2^2 - 4(-2)(3)}}{2(-2)} = \frac{2 \pm \sqrt{28}}{-4} = -\frac{2 \pm \sqrt{4}\sqrt{7}}{4} = -\frac{1 \pm \sqrt{7}}{2}$$

So we have

$$-2\sin^2 x - 2\sin x + 3 = \left(\sin x + \frac{1+\sqrt{7}}{2}\right)\left(\sin x + \frac{1-\sqrt{7}}{2}\right) = 0$$

For each factor, we try to find one solution using inverse sine. However $(1+\sqrt{7})2 > 1$ is out of the range of $\sin x$, and thus there cannot be a solution to $\sin x = -(1+\sqrt{7})/2$, so the only solutions will come from the $(\sin x + \frac{1-\sqrt{7}}{2})$ factor. We'll use inverse sine to find one, and for the other solution in $[0, 2\pi)$, we actually need to use the identity $\sin(\pi - x) = \sin x$ (this is a consequence of the identity for Sine of a Difference). Then all solutions will be

$$x = \begin{cases} \arcsin\left(-\frac{1-\sqrt{7}}{2}\right) + 2\pi n \\ \pi - \arcsin\left(-\frac{1-\sqrt{7}}{2}\right) + 2\pi n \end{cases}$$

If the last few steps of this example didn't make sense, that's ok. I didn't realize the problem would be this complicated when I assigned it, and you won't have anything this difficult on the final. I would just make sure you understand how to factor equations that are quadratic in a trig function, like in $\#2, 3$.

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