REVIEW EXERCISES AND PROBLEMS FOR CHAPTER ONE

Exercises

1. The entire graph of $f(x)$ is shown in Figure 1.87.

(a) What is the domain of $f(x)$?
(b) What is the range of $f(x)$?
(c) List all zeros of $f(x)$.
(d) List all intervals on which $f(x)$ is decreasing.
(e) Is $f(x)$ concave up or concave down at $x = 6$?
(f) What is $f(4)$?
(g) Is this function invertible? Explain.

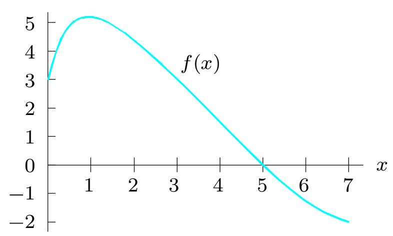


Figure 1.87

For Exercises 2–5, solve for x using logs.

2. $10 = 4^x$

3. $25 = 2(5)^x$

4. $2 \cdot 5^x = 11 \cdot 7^x$

5. $7 = 5e^{0.2x}$

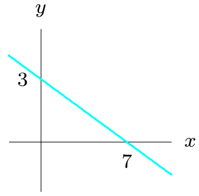
Find the amplitudes and periods in Exercises 6–7.

6. $y = 5\sin(x/3)$

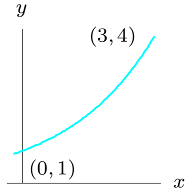
7. $y = 4 - 2\cos(5x)$

Find possible formulas for the graphs in Exercises 8–21.

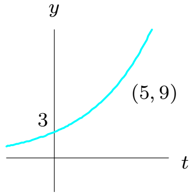
8.



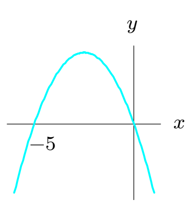
9.



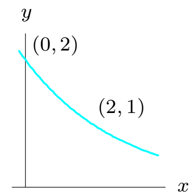
10.



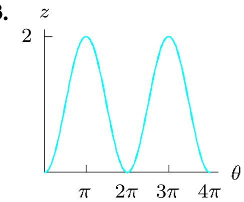
11.



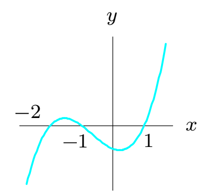
12.



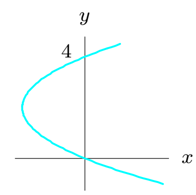
13.



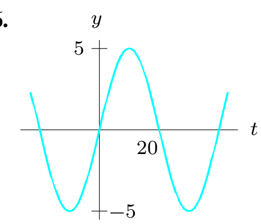
14.



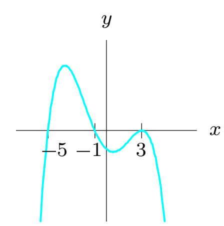
15.



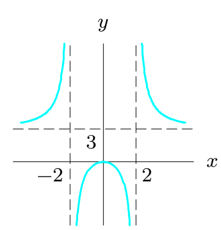
16.



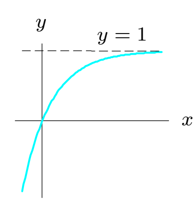
17.



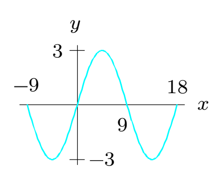
18.



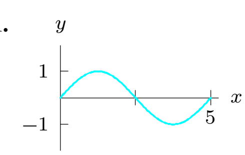
19.



20.



21.



For Exercises 22–23, find functions f and g such that $h(x) = f(g(x))$. [Note: Do not choose $f(x) = x$ or $g(x) = x$.]

22. $h(x) = \ln(x^3)$

23. $h(x) = (\ln x)^3$

State whether the functions in Exercises 24–27 are continuous on the interval $[-1, 1]$.

24. $f(x) = |x|$

25. $g(x) = \dfrac{|x|}{x}$

26. $h(\theta) = \theta \sin\theta$

27. $f(t) = \dfrac{\sin t}{t^2}$

For the functions in Exercises 28–31, do the following:

(a) Make a table of values of $f(x)$ for $x = a+0.1, a+0.01, a+0.001, a+0.0001, a-0.1, a-0.01, a-0.001$, and $a-0.0001$.

(b) Make a conjecture about the value of $\lim_{x\to a} f(x)$.

(c) Graph the function to see if it is consistent with your answers to parts (a) and (b).

(d) Find an interval for x containing a such that the difference between your conjectured limit and the value of the function is less than 0.01 on that interval. (In other words, find a window of height 0.02 such that the graph exits the sides of the window and not the top or bottom of the window.)

28. $f(x) = \dfrac{\cos 2x - 1 + 2x^2}{x^3}, \quad a = 0$

29. $f(x) = \dfrac{\cos 3x - 1 + 4.5x^2}{x^3}, \quad a = 0$

30. $f(x) = \dfrac{\sin 5x - 1}{x - \pi/2}, \quad a = \dfrac{\pi}{2}$

31. $f(x) = \dfrac{e^{0.5x-1} - 1}{x - 2}, \quad a = 2$

For the functions in Exercises 32–33, use algebra to evaluate the limits $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$, and $\lim_{x\to a} f(x)$ if they exist. Sketch a graph to confirm your answers.

32. $a = 3, \quad f(x) = \dfrac{x^3|2x-6|}{x-3}$

33. $a = 0, \quad f(x) = \begin{cases} e^x, & -1 < x < 0 \\ 1, & x = 0 \\ \cos x, & 0 < x < 1 \end{cases}$

Problems

34. The yield, Y, of an apple orchard (in bushels) as a function of the amount, a, of fertilizer (in pounds) used on the orchard is shown in Figure 1.88.

(a) Describe the effect of the amount of fertilizer on the yield of the orchard.
(b) What is the vertical intercept? Explain what it means in terms of apples and fertilizer.
(c) What is the horizontal intercept? Explain what it means in terms of apples and fertilizer.
(d) What is the range of this function for $0 \le a \le 80$?
(e) Is the function increasing or decreasing at $a = 60$?
(f) Is the graph concave up or down near $a = 40$?

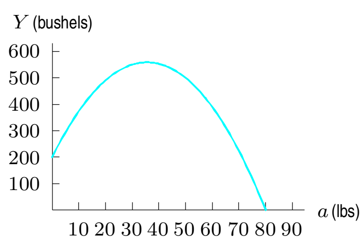


Figure 1.88

35. A rock is dropped from a window and falls to the ground below. The height, s (in meters), of the rock above ground is a function of the time, t (in seconds), since the rock was dropped, so $s = f(t)$.

(a) Sketch a possible graph of s as a function of t.
(b) Explain what the statement $f(7) = 12$ tells us about the rock's fall.
(c) The graph drawn as the answer for part (a) should have a horizontal and vertical intercept. Interpret each intercept in terms of the rock's fall.

36. A culture of 100 bacteria doubles after 2 hours. How long will it take for the number of bacteria to reach 3,200?

37. A controversial 1992 Danish study[9] reported that men's average sperm count has decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.

(a) Express the average sperm count, S, as a linear function of the number of years, t, since 1940.
(b) A man's fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?

38. Different kinds of the same element (called different *isotopes*) can have very different half-lives. The decay of plutonium-240 is described by the formula

$$Q = Q_0 e^{-0.00011t},$$

whereas the decay of plutonium-242 is described by

$$Q = Q_0 e^{-0.0000018t}.$$

Find the half-lives of plutonium-240 and plutonium-242.

39. A culture of bacteria originally numbers 500. After 2 hours there are 1500 bacteria in the culture. Assuming exponential growth, how many are there after 6 hours?

40. The population of the United States was 226.5 million in 1980 and 248.7 million in 1990. Assuming exponential growth, in what year is the population expected to go over 300 million?

[9]"Investigating the Next Silent Spring", *US News and World Report*, p. 50-52, (March 11, 1996).

41. One of the main contaminants of a nuclear accident, such as that at Chernobyl, is strontium-90, which decays exponentially at a continuous rate of approximately 2.47% per year. After the Chernobyl disaster, it was suggested that it would be about 100 years before the region would again be safe for human habitation. What percent of the original strontium-90 would still remain then?

42. In the early 1920s, Germany had tremendously high inflation, called hyperinflation. Photographs of the time show people going to the store with wheelbarrows full of money. If a loaf of bread cost $1/4$ RM in 1919 and 2,400,000 RM in 1922, what was the average yearly inflation rate between 1919 and 1922?

43. An airplane uses a fixed amount of fuel for takeoff, a (different) fixed amount for landing, and a third fixed amount per mile when it is in the air. How does the total quantity of fuel required depend on the length of the trip? Write a formula for the function involved. Explain the meaning of the constants in your formula.

44. Each planet moves around the sun in an elliptical orbit. The orbital period, T, of a planet is the time it takes the planet to go once around the sun. The semimajor axis of each planet's orbit is the average of the largest and the smallest distances between the planet and the sun. Johannes Kepler (1571-1630) discovered that the period of a planet is proportional to the $\frac{3}{2}$ power of its semimajor axis. What is the orbiting period (in days) of Mercury, the closest planet to the sun, with a semimajor axis of 58 million km? What is the period (in years) of Pluto, the farthest planet, with a semimajor axis of 6000 million km? The semimajor axis of the earth is 150 million km. [Hint: What is the earth's period?]

45. A closed cylindrical can of fixed volume V has radius r.

(a) Find the surface area, S, as a function of r.
(b) What happens to the value of S as $r \to \infty$?
(c) Sketch a graph of S against r, if $V = 10$ cm^3.

46. **(a)** Consider the functions graphed in Figure 1.89(a). Find the coordinates of C.
(b) Consider the functions in Figure 1.89(b). Find the coordinates of C in terms of b.

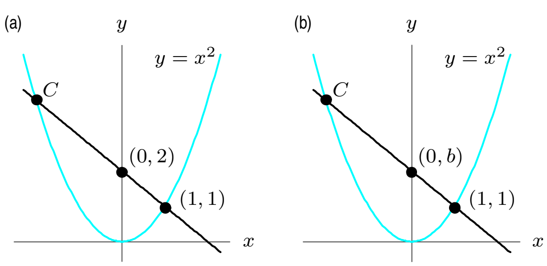


Figure 1.89

47. The depth of water in a tank oscillates sinusoidally once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.

48. The visitors' guide to St. Petersburg, Florida, contains the chart shown in Figure 1.90 to advertise their good weather. Fit a trigonometric function approximately to the data, where H is temperature in degrees Fahrenheit, and the independent variable is time in months. In order to do this, you will need to estimate the amplitude and period of the data, and when the maximum occurs. (There are many possible answers to this problem, depending on how you read the graph.)

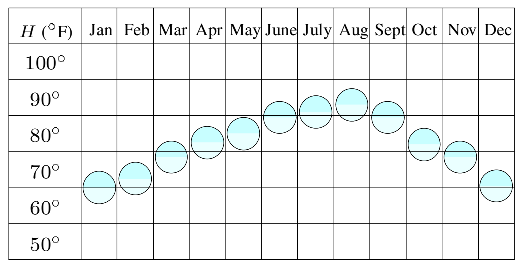


Figure 1.90: "St. Petersburg...where we're famous for our wonderful weather and year-round sunshine." (Reprinted with permission)

49. Figure 1.91 is the graph[10] of the function f, where $f(t)$ is the number (in millions) of motor vehicles registered in the world in the year t. (In 1988, one-third of the registered vehicles in the world were in the United States.)

(a) Is f invertible? Explain.
(b) What is the meaning of $f^{-1}(400)$ in practical terms? Evaluate $f^{-1}(400)$.
(c) Sketch the graph of f^{-1}.

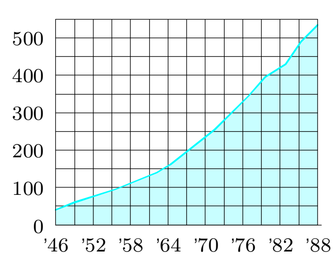


Figure 1.91

[10]From D. Blerics and P. Walzer, "Energy for Motor Vehicles," *Scientific American*, September 1990.

50. Each of the functions in the table is increasing over its domain, but each increases in a different way. Match the functions f, g, h to the graphs in Figure 1.92.

x	$f(x)$
1	1
2	2
4	3
7	4
11	5
16	6
22	7
29	8
37	9
47	10

x	$g(x)$
3.0	1
3.2	2
3.4	3
3.6	4
3.8	5
4.0	6
4.2	7
4.4	8
4.6	9
4.8	10

x	$h(x)$
10	1
20	2
28	3
34	4
39	5
43	6
46.5	7
49	8
51	9
52	10

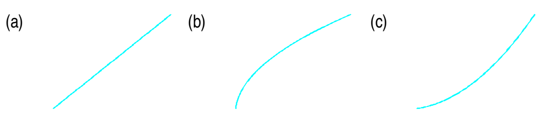


Figure 1.92

51. **(a)** Use a graphing calculator or computer to estimate the period of $2\sin\theta + 3\cos(2\theta)$.

(b) Explain your answer, given that the period of $\sin\theta$ is 2π and the period of $\cos(2\theta)$ is π.

52. Glucose is fed by intravenous injection at a constant rate, k, into a patient's bloodstream. Once there, the glucose is removed at a rate proportional to the amount of glucose present. If R is the net rate at which the quantity, G, of glucose in the blood is increasing:

(a) Write a formula giving R as a function of G.

(b) Sketch a graph of R against G.

53. A catalyst in a chemical reaction is a substance which speeds up the reaction but which does not itself change. If the product of a reaction is itself a catalyst, the reaction is said to be autocatalytic. The rate, r, of an autocatalytic reaction is proportional to the quantity of the original material remaining times the quantity of product, p, produced. The initial quantity of the original material is A and the amount remaining is $A - p$.

(a) Express r as a function of p.

(b) What is the value of p when the reaction is proceeding fastest?

54. What approximate domains and ranges make the graphs of $y = x^2$ and $y = 0.01e^{0.01x}$ look like each of the graphs in Figure 1.93?

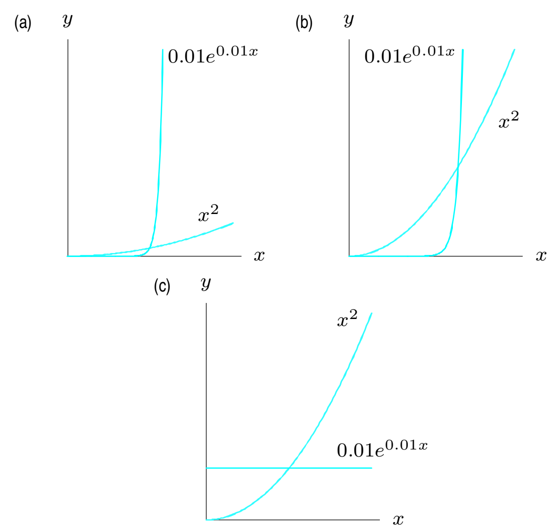


Figure 1.93

For each value of ϵ in Problems 55–56, find a positive value of δ such that the graph of the function leaves the window $a - \delta < x < a + \delta$, $b - \epsilon < y < b + \epsilon$ by the sides and not through the top or bottom.

55. $h(x) = \sin x$, $a = b = 0$, $\epsilon = 0.1, 0.05, 0.0007$.

56. $k(x) = \cos x$, $a = 0$, $b = 1$, $\epsilon = 0.1, 0.001, 0.00001$.

CAS Challenge Problems

57. **(a)** Factor $f(x) = x^4 + bx^3 - cx^3 - a^2x^2 - bcx^2 - a^2bx + a^2cx + a^2bc$ using a computer algebra system.

(b) Assuming a, b, c are constants with $0 < a < b < c$, use your answer to part (a) to make a hand sketch of the graph of f. Explain how you know its shape.

58. **(a)** Using a computer algebra system, factor $f(x) = -x^5 + 11x^4 - 46x^3 + 90x^2 - 81x + 27$.

(b) Use your answer to part (a) to make a hand sketch of the graph of f. Explain how you know its shape.

59. Let $f(x) = e^{6x} + e^{5x} - 2e^{4x} - 10e^{3x} - 8e^{2x} + 16e^x + 16$.

(a) What happens to the value of $f(x)$ as $x \to \infty$? As $x \to -\infty$? Explain your answer.

(b) Using a computer algebra system, factor $f(x)$ and predict the number of zeros of the function $f(x)$.

(c) What are the exact values of the zeros? What is the relationship between successive zeros?

60. Let $f(x) = x^2 - x$.

(a) Find the polynomials $f(f(x))$ and $f(f(f(x)))$ in expanded form.

(b) What do you expect to be the degree of the polynomial $f(f(f(f(f(f(x))))))$? Explain.

61. **(a)** Use a computer algebra system to rewrite the rational function
$$f(x) = \frac{x^3 - 30}{x - 3}$$
in the form
$$f(x) = p(x) + \frac{r(x)}{q(x)},$$
where $p(x)$, $q(x)$, $r(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $q(x)$.

(b) What is the vertical asymptote of f? Use your answer to part (a) to write the formula for a function whose graph looks like the graph of f for x near the vertical asymptote.

(c) Use your answer to part (a) to write the formula for a function whose graph looks like the graph of f for $x \to \infty$ and $x \to -\infty$.

(d) Using graphs, confirm the asymptote you found in part (b) and the formula you found in part (c).

We say that a function can be written as a polynomial in $\sin x$ (or $\cos x$) if it is of the form $p(\sin x)$ (or $p(\cos x)$) for some polynomial $p(x)$. For example, $\cos 2x$ can be written as a polynomial in $\sin x$ because $\cos(2x) = 1 - 2\sin^2 x = p(\sin x)$, where $p(x) = 1 - 2x^2$.

62. Use the trigonometric capabilities of your computer algebra system to express $\sin(5x)$ as a polynomial in $\sin x$.

63. Use the trigonometric capabilities of your computer algebra system to express $\cos(4x)$ as a polynomial in

(a) $\sin x$

(b) $\cos x$.

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–30 true or false? Give an explanation for your answer.

1. For any two points in the plane, there is a linear function whose graph passes through them.
2. The graph of $f(x) = 100(10^x)$ is a horizontal shift of the graph of $g(x) = 10^x$.
3. The graph of $f(x) = \ln x$ is concave down.
4. The graph of $g(x) = \log(x - 1)$ crosses the x-axis at $x = 1$.
5. Every polynomial of odd degree has at least one zero.
6. The function $y = 2 + 3e^{-t}$ has a y-intercept of $y = 3$.
7. The function $y = 5 - 3e^{-4t}$ has a horizontal asymptote of $y = 5$.
8. If $y = f(x)$ is a linear function, then increasing x by 1 unit changes the corresponding y by m units, where m is the slope.
9. If $y = f(x)$ is an exponential function and if increasing x by 1 increases y by a factor of 5, then increasing x by 2 increases y by a factor of 10.
10. If $y = Ab^x$ and increasing x by 1 increases y by a factor of 3, then increasing x by 2 increases y by a factor of 9.
11. The function $f(\theta) = \cos\theta - \sin\theta$ is increasing on $0 \le \theta \le \pi/2$.
12. The function $f(t) = \sin(0.05\pi t)$ has period 0.05.
13. If t is in seconds, $g(t) = \cos(200\pi t)$ executes 100 cycles in one second.
14. The function $f(\theta) = \tan(\theta - \pi/2)$ is not defined at $\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$.
15. The function $f(x) = \sin(x^2)$ is periodic, with period 2π.
16. The function $g(\theta) = e^{\sin\theta}$ is periodic.
17. If $f(x)$ is a periodic function with period k, then $f(g(x))$ is periodic with period k for every function $g(x)$.
18. If $g(x)$ is a periodic function, then $f(g(x))$ is periodic for every function $f(x)$.
19. The function $f(x) = e^{-x^2}$ is decreasing for all x.
20. The inverse function of $y = \log x$ is $y = 1/\log x$.
21. If f is an increasing function, then f^{-1} is an increasing function.
22. The function $f(x) = |\sin x|$ is even.
23. If a function is even, then it does not have an inverse.
24. If a function is odd, then it does not have an inverse function.
25. If a and b are positive constants, $b \ne 1$, then $y = a + ab^x$ has a horizontal asymptote.
26. If a and b are positive constants, then $y = \ln(ax + b)$ has no vertical asymptote.
27. The function $y = 20/(1 + 2e^{-kt})$ with $k > 0$, has a horizontal asymptote at $y = 20$.
28. If $g(x)$ is an even function then $f(g(x))$ is even for every function $f(x)$.
29. If $f(x)$ is an even function then $f(g(x))$ is even for every function $g(x)$.
30. If $\lim_{h\to 0} f(h) = L$, then $f(0.0001)$ is closer to L than is $f(0.01)$.

In Problems 31–36, give an example of a function with the specified properties. Express your answer using formulas.

31. Continuous on $[0, 1]$ but not continuous on $[1, 3]$.

32. Increasing but not continuous on $[0, 10]$.

33. Has a vertical asymptote at $x = -7\pi$.

34. Has exactly 17 vertical asymptotes.

35. Has a vertical asymptote which is crossed by a horizontal asymptote.

36. Two functions $f(x)$ and $g(x)$ such that moving the graph of f to the left 2 units gives the graph of g and moving the graph of f up 3 also gives the graph of g.

Suppose f is an increasing function and g is a decreasing function. In Problems 37–40, give an example for f and g for which the statement is true, or say why such an example is impossible.

37. $f(x) + g(x)$ is decreasing for all x.

38. $f(x) - g(x)$ is decreasing for all x.

39. $f(x)g(x)$ is decreasing for all x.

40. $f(g(x))$ is increasing for all x.

Are the statements in Problems 41–49 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

41. Every rational function that is not a polynomial has a vertical asymptote.

42. If a function is increasing on an interval, then it is concave up on that interval.

43. If y is a linear function of x, then the ratio y/x is constant for all points on the graph at which $x \neq 0$.

44. An exponential function can be decreasing.

45. If $y = f(x)$ is a linear function, then increasing x by 2 units adds $m + 2$ units to the corresponding y, where m is the slope.

46. If a function is not continuous at a point, then it is not defined at that point.

47. If f is continuous on the interval $[0, 10]$ and $f(x) = 0$ and $f(10) = 100$, then $f(c)$ cannot be negative for c in $[0, 10]$.

48. If $f(x)$ is not continuous on the interval $[a, b]$, then $f(x)$ must omit at least one value between $f(a)$ and $f(b)$.

49. There is a function which is both even and odd.

Suppose that $\lim_{x\to 3} f(x) = 7$. Are the statements in Problems 50–56 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

50. $\lim_{x\to 3}(xf(x)) = 21$.

51. If $g(3) = 4$, then $\lim_{x\to 3}(f(x)g(x)) = 28$.

52. If $\lim_{x\to 3} g(x) = 5$, then $\lim_{x\to 3}(f(x) + g(x)) = 12$.

53. If $\lim_{x\to 3}(f(x) + g(x)) = 12$, then $\lim_{x\to 3} g(x) = 5$.

54. $f(2.99)$ is closer to 7 than $f(2.9)$ is.

55. If $f(3.1) > 0$, then $f(3.01) > 0$.

56. If $\lim_{x\to 3} g(x)$ does not exist, then $\lim_{x\to 3}(f(x)g(x))$ does not exist.

Which of the statements in Problems 57–61 are true about every function $f(x)$ such that $\lim\limits_{x\to c} f(x) = L$? Give a reason for your answer.

57. If $f(x)$ is within 10^{-3} of L, then x is within 10^{-3} of c.

58. There is a positive ϵ such that, provided x is within 10^{-3} of c, and $x \neq c$, we can be sure $f(x)$ is within ϵ of L.

59. For any positive ϵ, we can find a positive δ such that, provided x is within δ of c, and $x \neq c$, we can be sure that $f(x)$ is within ϵ of L.

60. For each $\epsilon > 0$, there is a $\delta > 0$ such that if x is not within δ of c, then $f(x)$ is not within ϵ of L.

61. For each $\epsilon > 0$, there is some $\delta > 0$ such that if $f(x)$ is within ϵ of L, then we can be sure that x is within δ of c.

62. Which of the following statements is a direct consequence of the statement: "If f and g are continuous at $x = a$ and $g(a) \neq 0$ then f/g is continuous at $x = a$"?

(a) If f and g are continuous at $x = a$ and $f(a) \neq 0$ then g/f is continuous at $x = a$.
(b) If f and g are continuous at $x = a$ and $g(a) = 0$, then f/g is not continuous at $x = a$.
(c) If f, g, are continuous at $x = a$, but f/g is not continuous at $x = a$, then $g(a) = 0$.
(d) If f and f/g are continuous at $x = a$ and $g(a) \neq 0$, then g is continuous at $x = a$.