Solving Inequalities with Logarithms and Exponents

1.

$$\log_2\left(\frac{2x-1}{x-2}\right) < 0$$

Let's start by seeing which x can even be put into the logarithm function in the inequality. Since the domain of a logarithm function is $(0,\infty)$, we need

$$\frac{2x-1}{x-2} > 0$$

We know the fraction will be positive when both the numerator and denominator are the same sign. To see when each piece changes sign, we look for when each piece is 0. The numerator: $2x-1=0 \Rightarrow x=1/2$. The denominator: $x-2=0 \Rightarrow x=2$. We now divide up the real line into intervals divided by these numbers $x=1/2, 2$, and check the sign of the numerator and denominator in each interval.

Interval	$(-\infty,1/2)$	$(1/2,2)$	$(2,\infty)$
$2x-1$	-	+	+
$x-2$	-	-	+
$\frac{2x-1}{x-2}$	+	-	+

So the domain of $\log_2\left(\frac{2x-1}{x-2}\right)$ is $(-\infty,1/2)\bigcup(2,\infty)$. Note we do not include the end-points (why?).

Now lets go on to solve the inequality. We first exponentiate both sides with base 2 (since this is the inverse operation of logarithm base 2).

$$\frac{2x-1}{x-2} < 2^0 = 1$$

The next step in isolating x is to multiply by both sides by $x-2$. Since multiplying by a negative number reverses an inequality, **we need to be careful about the sign of $x-2$, since that will affect the direction of the inequality**. Let's first say $x>2$, so $x-2>0$ and the direction of the inequality doesn't change, and we get

$$\begin{aligned} 2x-1 &< x-2 \\ 2x-x &< 1-2 \\ x &< -1 \end{aligned}$$

Well this violates our assumption that $x>2$, so this tells us $(2,\infty)$ **is not part of our solution set**. Next say $x<2$, so $x-2$ is negative, and when we multiply by this we need to switch the sign of the inequality:

$$\begin{aligned} \frac{2x-1}{x-2} &< 1 \\ 2x-1 &> x-2 \\ x &> -1 \end{aligned}$$

So we have that $x > -1$ and our assumption here was $x < 2$, giving us x is in $(-1, 2)$. But now we need to go back and check the domain of the logarithm function, and see if any values in $(-1, 2)$ are not in the domain. The domain we found is $(-\infty, 1/2) \bigcup (2, \infty)$, so we see that $[1/2, 2)$ needs to be removed from $(-1, 2)$, leaving only $(-1, 1/2)$.

To illustrate this point, take $x = 1$. This satisfies the inequality $\frac{2x-1}{x-2} < 1$, but 1 is not in the domain of our original logarithm function and cannot be put into our original inequality.

So the final answer - the solution set to the inequality is $(-1, 1/2)$, or $-1 < x < 1/2$.

2.

$$\ln x + \ln(x-4) \leq \ln 21$$

Again we'll first check the domain. $\ln x$ gives us the domain restriction $x > 0$, and $\ln(x-4)$ gives us the domain restriction $x > 4$. Since a number > 4 is > 0, our domain is $x > 4$.

Now we'll solve the inequality, first using properties of logs and then exponentiating both side (with base e):

$$\begin{aligned} \ln x + \ln(x-4) &\leq \ln 21 \\ \ln(x(x-4)) &\leq \ln 21 \\ x(x-4) &\leq 21 \\ x^2 - 4x - 21 &\leq 0 \\ (x-7)(x+3) &\leq 0 \end{aligned}$$

The product $(x-7)(x+3)$ will be less than 0 when the factors have opposite sign. You can set up a similar table as in the first problem, and we see that the factors are opposite sign when $-3 < x < 7$. Now since the inequality is $\leq$, not a strict $<$, we include the endpoints -3 and 7 (since the left side is allowed to actually $= 0$).

Now checking the domain, we found $x > 4$, so the actual solution set reduces from $[-3, 7]$ to $(4, 7]$. Notice 4 is not included, since it is not in the domain of $\ln(x-4)$.

So the final answer - the solution set to the inequality is $(4, 7]$, or $4 < x \leq 7$.