

Inverse Trigonometric Functions

As mentioned in class, when dealing with inverse trig functions, it is especially important to know the definitions and some basic properties. First, here is a summary of what the notation means:

$$\begin{array}{lll}
 y = \sin^{-1} x = \arcsin x & \Leftrightarrow & \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\
 y = \cos^{-1} x = \arccos x & \Leftrightarrow & \cos y = x \text{ and } 0 \leq y \leq \pi \\
 y = \tan^{-1} x = \arctan x & \Leftrightarrow & \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}
 \end{array}$$

Since $\sin x$, $\cos x$, $\tan x$ are periodic functions, we know they are not one-to-one and thus cannot have inverse functions on their entire domain. We instead define $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ based on *restricted* domains, where the trig function is both one-to-one, and onto its range. Notice the relationship between the domain, restricted domain, and range of a pair of "inverse" functions in this summary table:

Function	Domain	Restricted Domain	Range
$\sin x$	$(-\infty, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
$\sin^{-1} x$	$[-1, 1]$	NA	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos x$	$(-\infty, \infty)$	$[0, \pi]$	$[-1, 1]$
$\cos^{-1} x$	$[-1, 1]$	NA	$[0, \pi]$
$\tan x$	$(-\infty, \infty) - \{\frac{\pi}{2} + \pi k\}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
$\tan^{-1} x$	$(-\infty, \infty)$	NA	$(-\frac{\pi}{2}, \frac{\pi}{2})$

The most useful fact about inverse functions is that the composition of a function and its inverse is the identity. With inverse trig functions, we had to restrict domains, so this will not always be true. For example, $\sin(\sin^{-1} x) = x$ for every x in the domain of $\sin^{-1} x$, but $\sin^{-1}(\sin \pi) = 0 \neq \pi$. So to summarize:

$$\begin{array}{ll}
 \sin(\sin^{-1} x) = x & \text{for every } x \text{ in the domain of } \sin^{-1} x \\
 \sin^{-1}(\sin x) = x & \text{for every } x \text{ in the } \textit{restricted} \text{ domain of } \sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \\
 \cos(\cos^{-1} x) = x & \text{for every } x \text{ in the domain of } \cos^{-1} x \\
 \cos^{-1}(\cos x) = x & \text{for every } x \text{ in the } \textit{restricted} \text{ domain of } \cos x : [0, \pi] \\
 \tan(\tan^{-1} x) = x & \text{for every } x \text{ in the domain of } \tan^{-1} x \\
 \tan^{-1}(\tan x) = x & \text{for every } x \text{ in the } \textit{restricted} \text{ domain of } \tan x : (-\frac{\pi}{2}, \frac{\pi}{2})
 \end{array}$$