Solving Trigonometric Equations

Suppose we are given an equation such as $\sin 2x = \sin x$, and we want to solve for x, i.e. find all x values that make this equation true. Since, in general $\arcsin(\sin x) \neq x$, we cannot simply take arcsin of both sides of the equation in an attempt to isolate x. Rather, the strategy is to use identities to re-write the equation in a way where it will be easier to determine x. This is best illustrated through examples.

Example 1 Solve $\sin(2x) = \sin x$. First find all solutions $x \in [0, 2\pi]$, and then find all solutions.

Solution: We start by re-writing $\sin(2x)$ using the double angle identity:

$$\begin{aligned} \sin(2x) &= \sin x \\ 2\sin x \cos x &= \sin x \end{aligned}$$

We may be tempted to divide each side by $\sin x$, but that would be a problem if $\sin x = 0$. So we have to subtract $\sin x$ from both sides, and get

$$\begin{aligned} 2\sin x \cos x - \sin x &= 0 \\ \sin x\,(2\cos x - 1) &= 0 \end{aligned}$$

We've now re-written the equation as two factors whose product is 0. We know the only way this can happen is if either factor is equal to 0. Looking at the first factor $\sin x$, we want to know for which values of $x \in [0, 2\pi]$ give us $\sin x = 0$. Taking arcsin of both sides will give us one answer:

$$\begin{aligned} \arcsin(\sin x) &= \arcsin 0 \\ x &= \arcsin 0 = 0 \end{aligned}$$

This gives us one solution $x = 0$, but we know there are others in $[0, 2\pi]$. Using the unit circle definition of $\sin x$, we also know that $\sin \pi = 0$, and $\sin 2\pi = 0$ (although we know this last one by periodicity: $\sin(0) = \sin(0+2\pi)$). So far we have three solutions in $[0, 2\pi]$; $x = 0, \pi, 2\pi$.

Now lets look at the second factor $2\cos x - 1$, set this equal to 0 and solve for x:

$$\begin{aligned} 2\cos x - 1 &= 0 \\ 2\cos x &= 1 \\ \cos x &= \frac{1}{2} \end{aligned}$$

Now we can use arccos to find one solution:

$$\begin{aligned} \arccos(\cos x) &= \arccos \frac{1}{2} \\ x &= \frac{\pi}{3} \end{aligned}$$

Referring back to the unit circle definition of $\cos x$, we know there is another x value in $[0, 2\pi]$ that gives us $\cos x = \frac{1}{2}$ that lies in the 4th quadrant, which we could solve using a special reference triangle. However, we can also use the evenness and periodicity of cosine: since $\cos(x) = \cos(-x)$, setting $x = \pi/3$, we have

$$\frac{1}{2} = \cos\left(\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3} + 2\pi\right) = \cos\left(\frac{5\pi}{3}\right)$$

So we have two more solutions $x = \pi/3, 5\pi/3$.

Then all solutions that lie in the interval $[0, 2\pi]$ to the equation $\sin(2x) = \sin x$ are

$$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$$

To get all solutions x, we now simply add integer multiples of 2π (the period of $\sin x$ and $\cos x$) to each solution:

$$x = \begin{cases} 0 + 2\pi n \\ \frac{\pi}{3} + 2\pi n \\ \pi + 2\pi n \\ \frac{5\pi}{3} + 2\pi n \end{cases}$$

where n is any integer (positive or negative). Note we haven't explicitly written 2π since it is just $0 + 2\pi n$ with $n = 1$. These are now *all* solutions to $\sin(2x) = \sin x$.

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Example 2 Solve $\sin(-x) = \cos x$. First find all solutions $x \in [0, 2\pi]$, and then find all solutions.

Solution: We again want to rewrite this using the identities we know. It would be nice if we could get this to involve just one trig function:

$$\begin{aligned} \sin(-x) &= \cos x \\ -\sin x &= \cos x \end{aligned}$$

Now we know that $\cos x$ and $\sin x$ are never 0 for the same value of x (graphically, you can see this since $\sin x$ and $\cos x$ never intersect the x-axis at the same point; using the unit circle definitions, there is no point on the unit circle whose horizontal and vertical coordinate are both 0). Then there cannot possibly be a solution to this equation when $\cos x = 0$. This means we can divide by $\cos x$ to get:

$$\begin{aligned} -\sin x &= \cos x \\ -\frac{\sin x}{\cos x} &= 1 \\ -\tan x &= 1 \\ \tan x &= -1 \end{aligned}$$

We can now take arctan of both sides to give us one solution in $(-\pi/2, \pi/2)$.

$$\begin{aligned} \arctan(\tan x) &= \arctan -1 \\ x &= -\frac{\pi}{4} \end{aligned}$$

Now if we use the unit circle definition of $\tan x$ to find another solution to $\tan x = -1$, which will be in the 2nd quadrant, we get $x = 3\pi/4$. But this is just $-\pi/4 + \pi = 3\pi/4$, so our other solution in $[0, 2\pi]$ is just π, the period of $\tan x$, away from our first solution. Therefore *all* solutions to the equation $\sin(-x) = \cos x$ are

$$x = -\frac{\pi}{4} + \pi n$$

where n is any integer. Again, here we are adding integer multiples of π, not 2π, to our solution because the trig function that gave us the solution was $\tan x$.

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Example 3 Solve $\cos^2\theta = \cos\theta + \sin^2\theta$.

Solution: First, we notice if we use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to rewrite $\sin^2\theta$ in terms of $\cos^2\theta$, then our equation will only involve (powers of) $\cos\theta$:

$$\begin{aligned} \cos^2\theta &= \cos\theta + \sin^2\theta \\ \cos^2\theta &= \cos\theta + 1 - \cos^2\theta \end{aligned}$$

Bringing everything over to one side, we have

$$\begin{aligned} \cos^2\theta - \cos\theta - 1 + \cos^2\theta &= 0 \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \end{aligned}$$

Now, let's call $\cos\theta = x$, just to see what this expression looks like written in terms of x:

$$2x^2 - x - 1 = 0$$

This is simply a quadratic equation, which we know how to factor by using the quadratic formula

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Plugging in $a = 2, b = -1, c = -1$, we get

$$x = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4}$$

Therefore we have $x = 1, -1/2$. We now use this to factor our quadratic equation in x:

$$2x^2 - x - 1 = (x - 1)\left(x + \frac{1}{2}\right) = 0$$

Now returning to $x = \cos\theta$, we have

$$(\cos\theta - 1)\left(\cos\theta + \frac{1}{2}\right) = 0$$

We are now in the same position as we were in the first example; we set each factor equal to 0, and find all solutions in the interval $[0, 2\pi]$. I'll omit some of the details here:

$$\cos\theta - 1 = 0 \Rightarrow x = 0, 2\pi \qquad \cos\theta + \frac{1}{2} = 0 \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Since we got the solutions from $\cos\theta$, we add integer multiples of cosine's period 2π to each solution to write down *all* solutions to $\cos^2\theta = \cos\theta + \sin^2\theta$:

$$\theta = \begin{cases} 0 + 2\pi n \\ \frac{2\pi}{3} + 2\pi n \\ \frac{4\pi}{3} + 2\pi n \end{cases}$$

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Summary

- When given a trigonometric equation, use identities to re-express the equation. The goal is to either write the equation in terms of a single trig function (with the same input) that is equal to some value, or write the equation as factors whose product is 0.
- Find all the solutions in one cycle or interval. This will depend on the problem, and may be explicitly given in the problem, but will typically be an interval such as $[0, 2\pi], (-\pi/2, \pi/2)$, etc. It may be useful to refer back to the unit circle definition of the trig functions, and draw reference triangles to find all solutions in the specified interval; you typically *cannot* rely only on using inverse trig functions.
- To find all solutions, not just ones that lie in a specified interval, then we add integer multiples of the appropriate period to the solutions we've already found. By appropriate, I mean we add $2\pi n$ if the function that gave us our solution was $\sin x$ or $\cos x$, and add πn if the function that gave us our solution was $\tan x$, where n can be any integer.