The Four Transformations for Sine, Cosine

For functions of the form

$$y = d + a\sin\left(b(x-c)\right), \quad y = d + a\cos\left(b(x-c)\right)$$

we have the following table to summarize our results:

Parameter	**Property**	**Effect on y=sin x, y=cos x**
d	$d =$ Average Value	Vertical shift up if $d > 0$ Vertical shift down if $d < 0$
a	$\|a\| =$ Amplitude	Vertical stretch if $\|a\| > 1$ Vertical shrink if $\|a\| < 1$ Reflection across average value line if $a < 0$
b	$\frac{2\pi}{b} =$ Period	Horizontal shrink if $b > 1$ Horizontal stretch if $0 < b < 1$
c		Horizontal shift right if $c > 0$ Horizontal shift left if $c < 0$

A few remarks:

- Note that the amplitude is always a positive number, so we have absolute value signs around a.
- Note that while a, c, d may be any numbers, we are only considering b to be positive, i.e. $b > 0$.
- For questions where you are asked to come up with a function based on a graph or information given, the value of c will depend on whether you are using a sine function or a cosine function.
- Note the parentheses in

  $$y = d + a\sin\left(b(x-c)\right), \quad y = d + a\cos\left(b(x-c)\right)$$

  particularly the $b(x-c)$ term. In order to use the information from this table, the function must be in this form. For example, the function

  $$y = -2\cos(4x+4)$$

  is not in this form, and we *cannot* conclude that the horizontal shift is 4. Rather, we must do some algebra to bring function into the right form

  $$\begin{aligned} y &= -2\cos(4x+4) \\ &= -2\cos\left(4(x+1)\right) \end{aligned}$$

  So we see in fact that the horizontal shift is 1.

Example 1

$$y = 10 - 5\cos\left(\frac{\pi}{6}(x-2)\right)$$

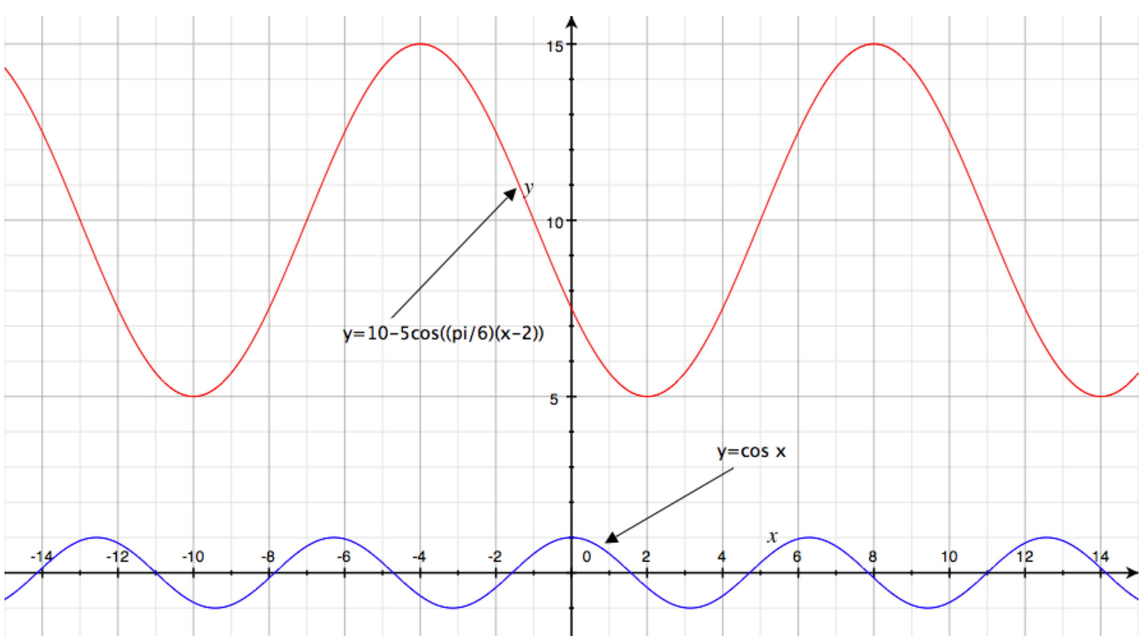
Solution: We'll use the table to extract information about the graph of the function

- Here $d = 10$, so there is a vertical shift up 10 units, and the average value of the function is 10.
- Here $a = -5$, so the amplitude is $|-5| = 5$. This means the maximum of the function is $10 + 5 = 15$, and the minimum of the function is $10 - 5 = 5$. Since there is a minus sign in front of the 5, we know there will be a reflection across the average value line $y = 10$.
- Here $b = \pi/6$, so the period of the function will be

$$p = \frac{2\pi}{\frac{\pi}{6}} = 2\pi\frac{6}{\pi} = 12$$

  So the period of the function is 12.

- The c value here is 2, so we know there is a horizontal shift to the right 2 units. Now at $x = 0, \cos x$ is at it's maximum. Then our function has a *minimum* at $x = 2$; the switch from maximum to minimum is because we have a reflection across the average value line. So we know $(2, 5)$ is a point on the graph. We know that if we move half a period either forward or backward on the x-axis, we will be at the function's maximum. Half a period is $12/2 = 6$, so we know two more points on the graph are $(-4, 15)$ and $(8, 15)$. We know the x values that correspond to the function's average value will be half way between the x values that correspond to a maximum and minimum value, so this gives us $(1, 10)$ and $(5, 10)$ are also points on the graph. We now have enough points to fill in the shape of the graph.



If we wanted to use a table to get the points on the graph that correspond to maximums, minimums, and average values, we would start with the following

$\mathbf{x}$	$\frac{\pi}{\mathbf{6}}(\mathbf{x}-\mathbf{2})$	$\mathbf{10}-\mathbf{5}\cos\left(\frac{\pi}{\mathbf{6}}(\mathbf{x}-\mathbf{2})\right)$
	0	
	$\frac{\pi}{2}$	
	π	
	$\frac{3\pi}{2}$	
	2π	

So our middle column is what the our function is inputting into cosine. We know what $\cos 0 = 1, \cos \pi/2 = 0$, etc. so we can use this to fill out the third column.

$\mathbf{x}$	$\frac{\pi}{\mathbf{6}}(\mathbf{x}-\mathbf{2})$	$\mathbf{10}-\mathbf{5}\cos\left(\frac{\pi}{\mathbf{6}}(\mathbf{x}-\mathbf{2})\right)$
	0	$5 = 10 - 5 \cdot 1$
	$\frac{\pi}{2}$	$10 = 10 - 5 \cdot 0$
	π	$15 = 10 - 5(-1)$
	$\frac{3\pi}{2}$	$10 = 10 - 5 \cdot 0$
	2π	$5 = 10 - 5 \cdot 1$

Now to find the first column, we need to take the middle row and solve for x. So for the first row, we solve $0 = \pi/6(x-2)$, which gives us $x = 2$. An easier way is to realize that each row is moving forward a quarter period, or $12/4 = 3$ units:

$\mathbf{x}$	$\frac{\pi}{\mathbf{6}}(\mathbf{x}-\mathbf{2})$	$\mathbf{10}-\mathbf{5}\cos\left(\frac{\pi}{\mathbf{6}}(\mathbf{x}-\mathbf{2})\right)$
2	0	5
5	$\frac{\pi}{2}$	10
8	π	15
11	$\frac{3\pi}{2}$	10
15	2π	5

And now we use the points from the first column plotted against the third column to get our graph.

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Example 2

(*This is #10 from Section 3.6 in the book*) On June 15th, the low temperature in Tucson was 75° F, and it occurred at 5:00 AM. The high temperature that day was 105°F, and it occurred at 5:00 PM. Assuming a sine or cosine provides a good model, find a possible formula for temperature as a function of time.

Solution: Let's start by putting what we know into a data table. We need to decide how to measure time, so let's set $t = 0$ to be 5:00 AM, so then t will be the time in hours after 5:00AM. Note here we are calling the dependent variable t instead of x.

t	Time	Temperature
0	5:00 AM	75°F (minimum value)
12	5:00 PM	105°F (maximum value)

So to put together a function, we can use the summary table at the beginning of this section to find the values for a, b, c, d. We'll go through in the order that they are listed on the table.

- The average value is the average between the maximum and minimum values, so $(105+75)/2 = 90$. Therefore our d value, or vertical shift, is $d = 90$.

- The amplitude is the distance between the average value and the maximum (or minimum) value, so $(105 - 90) = 15 = |a|$. We have to decide whether there should be a minus sign or not in front of the a; we'll leave this for later, as it will depend on our choice of horizontal shift and whether we're using sine or cosine.

- We're given that 12 hours are between the maximum and minimum temperature. We're assuming that these are the only high/low points of the day, so then this tells us that 12 is *half* the period. Therefore the period of our function is going to be 24 hours. We can now solve for b:
$$24 = \frac{2\pi}{b} \quad \Rightarrow b = \frac{\pi}{12}$$

- To determine what c value to use, we need to decide whether we are going to use sine or cosine in our formula. The way we have things set up, the function is reaching its minimum at $t = 0$. We know that $y = \cos t$ reaches its *maximum* at $t = 0$, so if we put a minus sign in front of the a amplitude, we will get a reflection across the average value line. Then if we use cosine, the function will reach its minimum at $t = 0$, which is exactly what we want. Then we do not need any horizontal shift, so $c = 0$.

Putting this all together, we have $a = -15, b = \frac{\pi}{12}, c = 0, d = 90$, and we are using a cosine function:
$$y = 90 - 15\cos\left(\frac{\pi}{12}t\right)$$

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How would you do this using a sine function? Try it.