# The Importance of a Story Line: Functions as Models of Change

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If students are to remember what they learn, the courses they take must tell a coherent story. This story provides a framework onto which they can hang their newly acquired knowledge. Without such a framework, teachers find themselves having to repeat material. Precalculus courses often run the risk of not being memorable because they are defined as the skills needed in calculus rather than telling a coherent story. Thus, the first decision in designing a new precalculus course is to choose the story it will tell.

The central theme we chose for our precalculus course is how functions can be used to model change.[1] This theme provides a framework into which all the prerequisites for calculus naturally fit (functions, graphing, algebra, trigonometry, numerical approximation), while at the same time illuminating a central concept of calculus—the rate of change. Choosing a family of functions to represent a real situation requires students to think about the qualitative behavior of different types of functions. A good way to decide for example, whether an exponential function fits a particular set of data is to look at a plot. The shape of the plot suggests the family; the values of the parameters are then determined from the data.

We have found that introducing the rate of change as the slope of a line is an excellent springboard for comparing the behavior of linear and exponential functions (absolute versus relative rate of change), and for introducing the concepts of increasing, decreasing, and concavity. Thus, even without the advantage of the derivative, pre-calculus students can experience some of the central ideas of calculus.

[1] *Functions Modeling Change: A Preparation for Calculus*, by Eric Connally et al, (New York: John Wiley, 2000).

No matter what flavor of calculus course they take, students benefit from a precalculus course that emphasizes interpretation as well as calculation. Particularly for students who are repeating material that they did not fully master before, a focus on meaning is an essential part of making the ideas fit together and finally stick. For example, the following problem asks students to think about the meaning of function notation:

> 1. The number of gallons of paint, $n$, needed to cover a house is a function of the surface area, $A$, measured in $\text{ft}^2$, of the house. That is, $n = f(A)$. Match each story below to one expression.
> (a) I figured out how many gallons I needed and then bought two extra gallons just in case.
> (b) I bought enough paint to cover my house twice.
> (c) I bought enough paint to cover my house and my welcome sign, which measures 2 square feet.
> (i) $2\ f(A)$ (ii) $f(A+2)$ (iii) $f(A)+2$

A pre-calculus course also needs to provide a context for reinforcing skills – though which skills are chosen may vary widely from instructor to instructor. Some will want to focus on algebraic and graphical fluency, others on the ability to model a real situation. Thus, we have put together a stock of varied problems. Problems which involve the use of parameters are particularly useful in our view. Working with functions expressed in terms of parameters is seldom familiar to students, and provides valuable experience both with algebraic manipulation and with understanding the behavior of an entire family of functions. As an example, consider the following problem:

> 2. Consider the exponential functions graphed in Figure 1 and the six constants $a, b, c, d, p, q$.
>
> (a) Which of these constants are definitely positive?
> (b) Which of these constants are definitely between 0 and 1?
> (c) Which of these constants could be between 0 and 1?
> (d) Which two of these constants are definitely equal?
> (e) Which one of the following pairs of constants could be equal?
> $a$ and $p$ $b$ and $d$ $b$ and $q$ $d$ and $q$

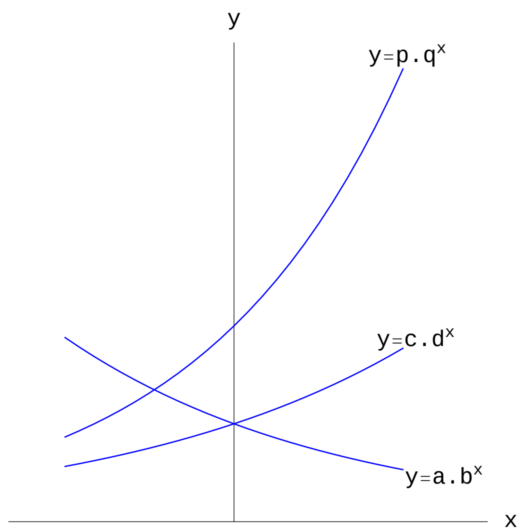


FIGURE 1

As in calculus, we believe students should be encouraged to make a connection between their calculations and reality. Since many students benefit from the experience of solving longer problems, we have written both problems and projects; an example of each follows.

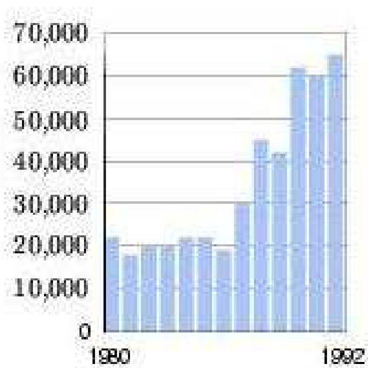


FIGURE 2

3. Hong Kong shifted from British to Chinese rule in 1997. Figure 2 shows the number of people who emigrated from Hong Kong during each of the years from 1980 to 1992.

(a) Find an exponential function that approximates the data.
(b) What does the model predict about the number of emigrants in 1996?
(c) Write a short paragraph explaining why this model is or is not useful to predict emigration in the year 2000.

4. The table shows the population of Ireland at various times between 1780 and 1910.

Population of Ireland, 1780-1910, where 0 corresponds to 1780.

| Years since 1780 | 0 | 20 | 40 | 60 | 70 | 90 | 110 | 130 |
|---|---|---|---|---|---|---|---|---|
| Population (millions) | 4.0 | 5.2 | 6.7 | 8.3 | 6.9 | 5.4 | 4.7 | 4.4 |

(a) When was the population increasing? Decreasing?
(b) For each successive time interval, construct a table showing the average rate of change of the population.
(c) From the table you constructed in part (b), when is the graph of the population concave up? Concave down?
(d) When was the average rate of change of the population the greatest? The least? How is this related to part (c)? What does this mean in human terms?
(e) Graph the data in the table and join the points by a curve to show the trend in the data. From this graph, identify where the curve is increasing, decreasing, concave up and concave down. Compare your answers to those you got in parts (a) and (c). Identify the region you found in part (d).
(f) Something catastrophic happened in Ireland between 1780 and 1910. When? What happened in Ireland at that time to cause this catastrophe?

We have successfully used these materials for several years both at the college and at the high school level. Some of the high school students using the materials were sufficiently enthusiastic that they made a presentation to the School Board about the mathematics they were learning. Since then, many have gone on to do excellent work in AP and college calculus.