# THE EFFECT OF TECHNOLOGY ON THE TEACHING OF MATHEMATICS

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The teaching of mathematics has been a source of particular concern in the US for the past 15 years. One of the problems is exemplified by the study in which school children were given the following question:

> A shepherd has 125 sheep in his flock and 5 dogs. How old is the shepherd?

Many children give a numerical answer (usually 25, obtained by observing that $125 + 5 = 130$ is too old, as is $125\text{-}5 = 120$, but that $125/5 = 25$ is about right). More alarming than the number of numerical answers, is the fact that children in older grades give numerical answers more frequently than younger children – making us wonder about the effectiveness of schooling.

On a local level, there are many occasions in my own teaching which suggest that students come to the university viewing mathematics as a subject in which they should be told what method to use. For example, the following problem was given on a calculus exam:

> A particle starts at the origin and moves along the graph of $y = x^2/2$ at a velocity of 10 units/second.
>
> (a) Write down the integral which shows how far the particle has travelled when it reaches the point where $x = a$.
>
> (b) You want to find the $x$-coordinate of the point reached by the particle after travelling for 2 seconds. Find upper and lower estimates, differing by less than 0.2, for this coordinate.

Some students described the problem as unfairly "vague," because it did not contain the words "arc length" to tell them what to do.

Surveys of Harvard students at the start of calculus and precalculus courses confirm that many believe math courses are about learning how to do computations rather than figuring out which computations to do. For example, in a survey of several hundred undergraduates, the question

> A well-written problem makes it clear what method to use to solve it.

received 4.1 out of 5 agreement from calculus students and 4.6 out of 5 agreement from precalculus students.

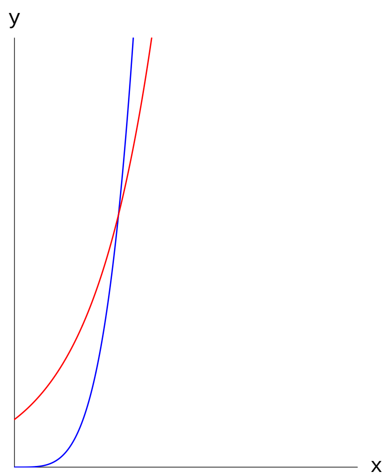
This attitude towards mathematics is an enormous handicap for students. It will block their chances of being successful in mathematics, pure or applied, and it will prevent them from using mathematics in any substantial way in other fields. If we are to produce mathematically able students and engineers, it is of the greatest importance that we address this problem.

Over the last 15 years there have been widespread efforts throughout the world to upgrade the curriculum at the high school and beginning college level. A study of examinations given 15 years ago shows that it was perfectly possible to pass and even do well in a calculus course by knowing how to calculate derivatives and integrals and without knowing what a derivative and integral meant. Questions like the following, which demand interpretation, have now been added to courses at many institutions:

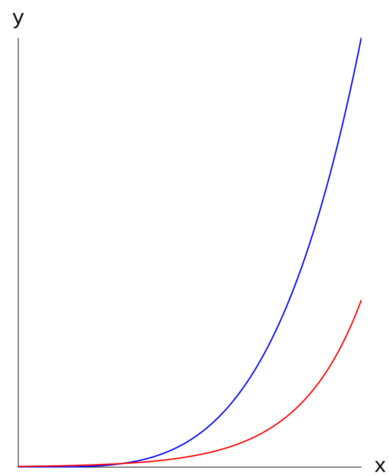
> 1. The US population (in millions) in year $t$ is represented by the function $P(t)$.
>
> (a) What do the functions $P(t) + 5$ and $P(t + 100)$ represent? (b) What do the statements $P'(1990) = 2.3$ and $(P^{-1})'(250) = 0.5$ tell us? What are the units of each derivative?
>
> 2. If $g(v)$ is the fuel efficiency of a car going at $v$ kilometers per hour (i.e., $g(v) =$ the number of km per liter at $v$ km/hr), what are the units of $g'(55)$? What is the practical meaning of the statement $g'(90) = -0.23$?

Technology – first graphing calculators and now computer algebra systems – have played, and continue to play, a significant role in challenging mathematics departments to make their courses thought-provoking. Graphing calculators forced us to rethink the teaching of graphing. With a calculator, producing the graph of a given function in a given window is easy. Focussing the course on how to obtain such graphs without using the calculator looks pointless to students. However, the following problem requires the students to think about the behavior of $x^4$ and $3^x$:

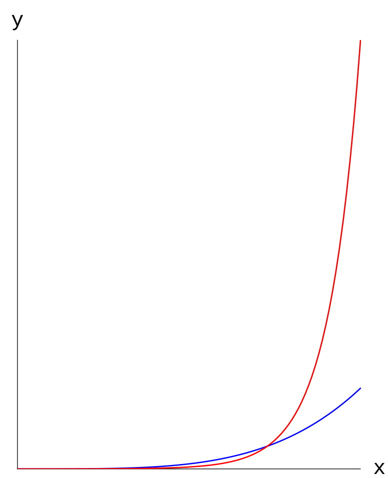
> Use a graphing calculator or a computer to graph $y = x^4$ and $y = 3^x$. Determine approximate domains and range that give each of the graphs in Figure 1.




(a)




(b)




(c)

FIGURE 1. $y = x^4$ vs. $y = 3^x$.

Attempts to do this problem by trial-and-error, without thinking about the mathematics, are doomed to failure. It requires the student to consider which function is largest where, how many times they intersect, and so on.

A calculator or computer can enable students to make the connection between different representations: the graphical, the numerical, and the symbolic; such connections are an important first step toward conceptual understanding. For example, consider the graph in Figure 2 of the parametric equations $x = \cos(t^3), y = \sin(t^3)$ for $0 \le t \le 10$ with $\Delta t = 0.1$

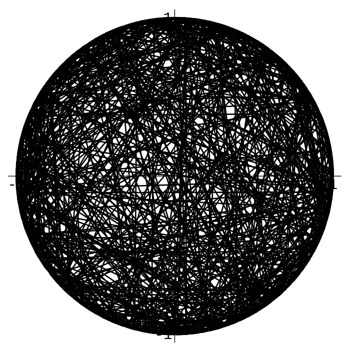


FIGURE 2. Graph for $x = \cos(t^3), y = \sin(t^3)$ for $0 \le t \le 10$ with $\Delta t = 0.1$

When I saw this shown to a class, they were at first surprised. Then they realized, by doing the algebra, that the graph really should be a circle. Then they figured out how the cube and the step size led to the picture they saw.

The use of calculators and computer can lead to the introduction of a more experimental, conjecture-based approach. This is more similar to real mathematics than the purely computational emphasis of the recent past. For example, the graph of a saddle point $z = x^2 - y^2$ caused my students to ask what function had the graph of a monkey saddle or a dog saddle. Using the 3-D graphing feature, they figured out the answer and then checked that they were right.

The fact that a graphical approach is easier with calculators and computers enables us to introduce some difficult topics earlier. For example, the traditional approach to introducing the function $e^x$ (by defining $\ln(x)$ as the integral $\int_1^x \frac{dt}{t}$ and $e^x$ as the inverse function of $\ln(x)$) has the effect of delaying $e^x$ till second semester calculus – long after the biologists would like to use it. This approach also involves several of the things students find most difficult (defining a function as an integral, logarithms, and inverse functions). In addition, it obscures the reason that $e^x$ is important: that the derivative of $e^x$ is $e^x$. The advantages of the traditional approach is that each of the definitions and results can be made rigorous. However, for many students, it may be better to delay the rigor and concentrate first on developing understanding. An approach which can be done earlier and which gives students a better sense of why $e^x$ is important is to study the function $f(x) = a^x$ and its derivative using the approximation $f'(x) \approx \frac{a^{x+0.01} - ax}{0.01}$, for various values of $a$. Graphs of $f(x)$ and $f'(x)$ for $a = 2$ and $a = 3$ are shown in Figure 3.

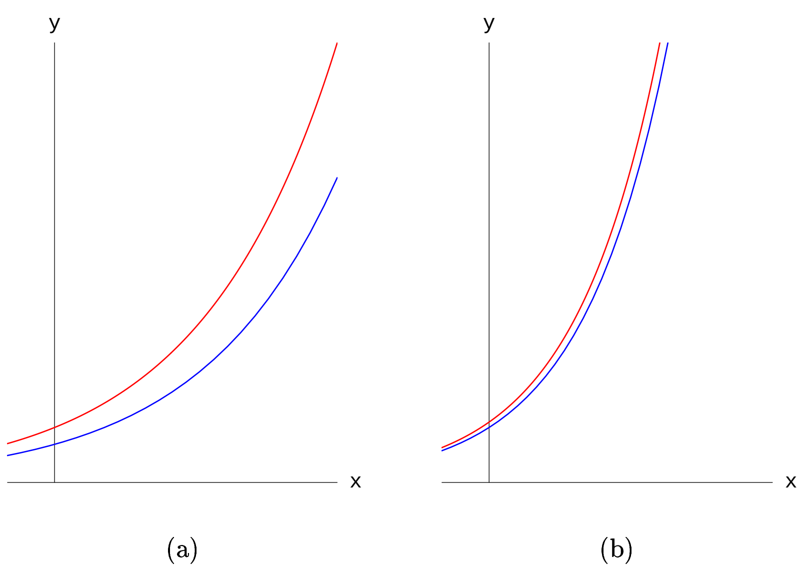



FIGURE 3. $f(x) = a^x$ and $f'(x)$ for (a) $a = 2$ and (b) $a = 3$.

The graph of $f(x)$ and $f'(x)$ certainly look the same shape. To suggest to students that $f(x)$ and $f'(x)$ are proportional to one another, look at the values in the table which shows $f(x), f'(x)$, and the ratio $f'(x)/f(x)$. The fact that the values of the ratio are constant (to three decimal places) suggests that $f'(x)$ and $f(x)$ are indeed proportional. This is confirmed by using the symbolic capability of a calculator or computer or by the usual limit calculation. The number $e$ is then introduced as the base which makes the derivative of $a^x$ equal to itself.

| $x$ | $f(x) = 2^x$ | $f'(x)$ | $f'(x)/f(x)$ | $g(x) = e^x$ | $g'(x)$ | $g'(x)/g(x)$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0.69556 | 0.69556 | 1 | 1.1047 | 1.1047 |
| 1 | 2 | 1.3911 | 0.69556 | 3 | 3.314 | 1.1047 |
| 2 | 4 | 2.7822 | 0.69556 | 9 | 9.942 | 1.1047 |
| 3 | 8 | 5.5644 | 0.69556 | 27 | 29.826 | 1.1047 |
| 4 | 16 | 11.129 | 0.69556 | 81 | 89.478 | 1.1047 |
| 5 | 32 | 22.258 | 0.69556 | 243 | 268.43 | 1.1047 |
| 6 | 64 | 44.516 | 0.69556 | 729 | 805.3 | 1.1047 |

Such a change in approach raises many questions. A central issue is the virtue or danger of having students gain an intuitive appreciation of the derivative of $a^x$ before they can prove the results. As you consider this question, it is sobering to realize how few can remember (never mind prove) the sequence of steps which lead to the traditional definition of $e^x$.

**Questions for the Future**

The changes that we have seen in the teaching of mathematics over the last 15 years foreshadows the much larger changes which will come when computer algebra systems become a regular part of schools. Since computer algebra systems are already available on easy to use calculators which cost less than US $200, and prices are falling, they could become common place in the next few years. If computer algebra systems are not to undermine the teaching of algebra – factoring, simplifying, solving equations – we must rethink the curriculum. To convince students that they should learn any particular manipulation by hand, we need to have a well thought-out argument for it. Similarly, any topic for which we can find no such argument is eligible to be dropped. The same issues arise in calculus: What do students need to be able to differentiate or integrate by hand, if they have a computer algebra system which can do this for them?

These questions are far harder and far more urgent than most people realize. The consequence of not answering them thoughtfully is that the decision may drift out of the hands of academics. Our answers to these questions will affect all the student who come to the university and therefore are of the greatest importance to all of us.