The isoperimetric problem in surfaces of revolution

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The isoperimetric problem in a surface

For a given surface $M \subset \mathbb{R}^3$, we look for the least-perimeter set in $M$ enclosing a fixed quantity of area.
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$\rightsquigarrow$ Isoperimetric regions
The isoperimetric problem in a surface

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\[ \implies \text{Isoperimetric regions} \]

- Existence
The isoperimetric problem in a surface

For a given surface $M \subset \mathbb{R}^3$, we look for the least-perimeter set in $M$ enclosing a fixed quantity of area

$\rightsquigarrow$ Isoperimetric regions

- Existence
- Bounded by closed embedded curves with constant geodesic curvature
The isoperimetric problem in a surface

For a given surface $M \subset \mathbb{R}^3$, we look for the least-perimeter set in $M$ enclosing a fixed quantity of area

\[\rightsquigarrow \text{Isoperimetric regions}\]

- Existence
- Bounded by closed embedded curves with constant geodesic curvature
- Classified for some surfaces
The isoperimetric problem in a surface

- Plane: Disks
The isoperimetric problem in a surface

- Plane: Disks
- Sphere: Geodesic disks
The isoperimetric problem in a surface

- Plane: Disks
- Sphere: Geodesic disks
- Right cylinder: Geodesic disks and horizontal strips
The isoperimetric problem in a surface

- Plane: Disks
- Sphere: Geodesic disks
- Right cylinder: Geodesic disks and horizontal strips
- Paraboloid: Geodesic disks centered at the origin

I. Benjamini and J. Cao, 1996
The isoperimetric problem in a surface

- Plane: Disks
- Sphere: Geodesic disks
- Right cylinder: Geodesic disks and horizontal strips
- Paraboloid: Geodesic disks centered at the origin
- Planes and spheres with monotonic Gauss curvature

F. Morgan, M. Hutchings and H. Howards, 2000; M. Ritoré, 2001
Our work

We study the isoperimetric problem in:

- symmetric **tori of revolution** with decreasing Gauss curvature
- symmetric **annuli of revolution** with increasing Gauss curvature
Our work

- Symmetric tori of revolution with decreasing Gauss curvature:

Standard torus of revolution
Our work

- Symmetric **annuli of revolution** with increasing Gauss curvature:

\[ S^1 \times \{0\} \]

*One-sheeted hyperboloid and catenoid*
Our approach

- Isoperimetric regions are bounded by constant geodesic curvature curves
- Isoperimetric regions are stable regions
Our approach

- Isoperimetric regions are bounded by constant geodesic curvature curves
- Isoperimetric regions are stable regions

$\Rightarrow$ isoperimetric candidates
Constant geodesic curvature curves

$M \subset \mathbb{R}^3$ surface of revolution

We will see $M$ as a warped product $S^1 \times I$ with metric

$$ds^2 = f(t)^2 d\theta^2 + dt^2,$$

where $I \subset \mathbb{R}$ is a real interval, and $f : I \to \mathbb{R}^+$ is a $C^1$ real function.
$M \subset \mathbb{R}^3$ surface of revolution

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where $I \subset \mathbb{R}$ is a real interval, and $f : I \to \mathbb{R}^+$ is a $C^1$ real function.

\[\leadsto \text{Classification of the curves in } M \text{ with constant geodesic curvature}\]
Constant geodesic curvature curves

- Circles of revolution: $S^1 \times \{t\}, \ t \in I$
Constant geodesic curvature curves

- Circles of revolution: $S^1 \times \{t\}, \ t \in I$
- Nodoids (bounding disks when closed)
Constant geodesic curvature curves

- Circles of revolution: $S^1 \times \{t\}, \ t \in I$
- Nodoids (bounding disks when closed)
- Unduloids (in general, not closed and embedded)
Annuli with increasing Gauss curvature

In symmetric annuli of revolution with increasing Gauss curvature:

- **Existence** of isoperimetric regions is not guaranteed (catenoids)
Annuli with increasing Gauss curvature

In symmetric annuli of revolution with increasing Gauss curvature:

- Closed embedded curves with constant geodesic curvature:
  - circles of revolution
  - nodoids
  - unduloids
Annuli with increasing Gauss curvature

The stable regions are

i) disks bounded by nodoids with constant Gauss curvature
Annuli with increasing Gauss curvature

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i) disks bounded by nodoids with constant Gauss curvature

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)
Annuli with increasing Gauss curvature

The **stable regions** are

i) disks bounded by nodoids with constant Gauss curvature

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an **unduloid** and a **circle of revolution**
Annuli with increasing Gauss curvature

The stable regions are

i) disks bounded by nodoids with constant Gauss curvature

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

iv) unions of a disk and a symmetric annulus
Annuli with increasing Gauss curvature

Finally, the *isoperimetric regions* are

i) disks with constant Gauss curvature (equal to its maximum)

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution
Annuli with increasing Gauss curvature

Isoperimetric regions in symmetric annuli of revolution with increasing Gauss curvature
Tori with decreasing Gauss curvature

We can see the torus as a warped product (finite annulus):
Torri with decreasing Gauss curvature

We can see the torus as a warped product (finite annulus):

\[ M = \mathbb{S}^1 \times [-t_0, t_0], \quad ds^2 = f(t)^2 d\theta^2 + dt^2 \]

By identifying \( \mathbb{S}^1 \times \{-t_0\} \) and \( \mathbb{S}^1 \times \{t_0\} \)

→ Torus of revolution
Tori with decreasing Gauss curvature

In symmetric tori of revolution with decreasing Gauss curvature:

- **Existence** of isoperimetric regions is **guaranteed** by compactness
In symmetric tori of revolution with decreasing Gauss curvature:

- Closed embedded curves with constant geodesic curvature:
  - circles of revolution
  - nodoids
  - unduloids
  - vertical geodesics
  - helix type curves
Tori with decreasing Gauss curvature

- **Vertical geodesics:**

  Generating curves of the torus of revolution
Tori with decreasing Gauss curvature

- Helix type curves:
  Geodesics in the torus (not closed in general)

Two different helix type curves in $[0, 2\pi] \times [-t_0, t_0]$
Tori with decreasing Gauss curvature

The **stable regions** are

i) **disks** bounded by nodoids
   (symmetric or with constant Gauss curvature)
Tori with decreasing Gauss curvature

The stable regions are

i) disks bounded by nodoids

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)
Tori with decreasing Gauss curvature

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Tori with decreasing Gauss curvature

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iv) unions of a disk and a symmetric annulus
Tori with decreasing Gauss curvature

The stable regions are

i) disks bounded by nodoids

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

iv) unions of a disk and a symmetric annulus

v) unions of vertical annuli (bounded by vertical geodesics)
Tori with decreasing Gauss curvature

The **stable regions** are

i) disks bounded by nodoids

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

iv) unions of a disk and a symmetric annulus

v) unions of vertical annuli

vi) unions of annuli bounded by helix type curves
Tori with decreasing Gauss curvature

Finally, the isoperimetric regions are

i) disks bounded by nodoids

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

iv) vertical annuli bounded by two vertical geodesics

v) unions of a disk and a symmetric annulus
Tori with decreasing Gauss curvature

Isoperimetric regions in symmetric tori of revolution with decreasing Gauss curvature
Main consequences

- **Unduloids** may appear in the isoperimetric boundaries
Main consequences

- Unduloids may appear in the isoperimetric boundaries
- The Gauss curvature of the surfaces may be piecewise continuous