Solutions to Homework 3, Math 575A Fall 2008

5.2 Consider an arbitrary $m \times n$ matrix $A$. Such a matrix has full rank if and only if all of its singular values are positive. Decompose $A = U \Sigma V^*$. Construct a sequence of matrices $A_k = U \Sigma_k V^*$, where $\Sigma_k$ is a diagonal matrix with diagonal entries $\sigma_j + 1/k$. Since its singular values are positive (they are non-negative, so adding $1/k$ makes for positive values), all we need to show is that the sequence has limit $A$, i.e. $||A_k - A||_2 \to 0$. $A_k - A = UDV^*$ here $D$ is a diagonal $m \times n$ matrix with diagonal entries $1/k$. The norm is that of $D$, i.e. $1/k$, which converges to 0.

5.3

a) $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 10\sqrt{2} & 0 \\ 0 & 10/\sqrt{2} \end{pmatrix}$, $V = \begin{pmatrix} -0.6 & 0.8 \\ 0.6 & 0.6 \end{pmatrix}$.

b) Sorry, no picture. Left singular vectors are the columns of $U$, right the columns of $V$, singular values are the diagonals of $\Sigma$.

c) $||A||_1 = \max ||a_j||_1 = 16$ (Using equation 3.9)

$||A||_\infty = \max ||a_i^*||_1 = 15$ (using equation 3.10, and the max over rows).

$||A||_2 = \max \sigma_k = 10\sqrt{2}$

$||A||_F = \sqrt{2^2 + 11^2 + 10^2 + 5^2} = \sqrt{250} = 5\sqrt{10}$.

d) $A^{-1} = (U \Sigma V^*)^{-1} = V \Sigma^{-1} U^* = \begin{pmatrix} 0.05 & -0.11 \\ 0.1 & -0.02 \end{pmatrix}$. The point is that it is easy to invert the unitary matrices, just take the adjoint, and the diagonal matrix is inverted by inverting the diagonal entries.

e) The eigenvalues are $(3 \pm \sqrt{391})/2$. (The eigenvalues are the roots of the characteristic polynomial $\det(\lambda I - A) = \lambda^2 - 3\lambda + 100 = 0$)

f) $\det(A) = 100$, $(3/2 + i\sqrt{391}/2)(3/2 - i\sqrt{391}/2) = 100$.

$\det(A)| = 100$ and $(10/\sqrt{2})(10/\sqrt{2}) = 100$.

$g)$ The area of an ellipse is $\pi$ times the product of the length of the semi-major axes. The lengths of the semi-major axes are the singular values. Thus the area is $100\pi$. We also know this since the absolute value of the determinant gives the factor by which area (volume) is multiplied by a linear transformation, and the area of the unit disk is $\pi$.

5.4 If $A = U \Sigma V^*$ is a singular value decomposition of a square matrix $A$, then we also have $A^* = V \Sigma U^*$, and so, we know that the singular values and vectors satisfy $A v_j = \sigma_j u_j$ and $A^* u_j = \sigma_j v_j$ for $j = 1, \ldots, m$. Thus we also can see that

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \sigma_j \begin{pmatrix} u \\ v \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} u \\ -v \end{pmatrix} = -\sigma_j \begin{pmatrix} u \\ -v \end{pmatrix}.$$ 

This shows that we have $2m$ independent eigenvectors with eigenvalues $\pm \sigma_j$.

This problem could also have been done by conjugating by the block matrix $\begin{pmatrix} V^* & 0 \\ 0 & U^* \end{pmatrix}$ to get a matrix in the form $\begin{pmatrix} 0 & \Sigma \\ \Sigma & 0 \end{pmatrix}$. That matrix can then be conjugated by $\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ to get a diagonal matrix with entries $\pm \sigma_j$. 

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6.1 To check that \( A = I - 2P \) is unitary, we use that \( P = P^* \) and that \( P^2 = P \) and calculate

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Geometrically, we are subtracting the projection of a vector \( v \), \( Pv \) from its complementary projection \( (I - P)v \). This is a reflection through the range of \( I - P \) (which is the null space of \( P \)). Note that \( (I - 2P)^2 = I \) as expected with a reflection, and of course it is unitary so it preserves angles and lengths.

6.2 The matrix for \( F \) has columns \( f_j = e_{m-j+1} \) where \( e_j \) are the columns of the identity matrix, that is the standard basis. It is clear that \( F^2 = I \), so to check if \( E = (I + F)/2 \) is a projector, look at \( E^2 = (I + 2F + F^2)/4 = (I + F)/2 = E \). To check if it is an orthogonal projector, we can use Theorem 6.1 and note that \( F^* = F \) so that \( E^* = E \) and thus \( E \) is an orthogonal projection. The \( ij^{th} \) entry of \( E \) is given by \( \delta_{i,j}/2 + \delta_{i,m-i+1}/2 \).

6.3 If \( A \in \mathbb{C}^{m \times n} \) with \( m \geq n \), then we can write \( A = \hat{U}\hat{\Sigma}V^* \) the reduced singular value decomposition of \( A \) with \( \hat{\Sigma} \) a \( n \times n \) diagonal matrix with diagonal entries \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \). \( A^*A = V\hat{\Sigma}^2V^* \). Since the \( n \) columns of \( \hat{U} \) are orthonormal, \( \hat{U}^*\hat{U} = I_n \), and \( A^*A = V\hat{\Sigma}^2V^* \). Thus the singular values (and eigenvalues) of \( A^*A \) are \( \sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_n^2 \). \( A^*A \) is nonsingular if and only if all of its singular values are positive, and this is true if and only if all of the singular values of \( A \) were positive. All singular values being positive is equivalent to maximal rank.