Problem 1 (10 points): Add comments to the Matlab function listed below, and explain what the function does, and how the lines in the code achieve that result. There are also some bugs in this program. Find them as well.

```matlab
function [W,R]=mystery(A)

[m,n]=size(A);
W=zeros(m,n);
R=A;
for k=1:m
    x=R(k:m,k);
    s=sign(x(1));
    if(s == 0)
        s=1;
    end
    v=x;
    v(1)=s*norm(x)+v(1);
    v=v/norm(v);
    W(k:m,k)=v;
    R(k:m,k:n)=R(k:m,k:n)-2*v'*v*R(k:m,k:n);
end
R=R(1:n,1:n);
```

Problem 2 (20 points): Suppose that a floating point computer satisfies equations (13.5) and (13.7) (axioms of floating point arithmetics), and further suppose that the square root function \( \sqrt{\cdot} \) is calculated in such a way as to satisfy

\[
\sqrt{x} = \sqrt{x}(1 + \epsilon), \quad \text{with } |\epsilon| \leq \epsilon_{\text{machine}}.
\]

For each of the following algorithms, indicate whether it is backward stable, stable or unstable:

1. input: \( x \) and \( y \); output: \( \sqrt{x \otimes x + y \otimes y} \)
2. input: \( x \); output: \( \sqrt{1 + x \otimes x} \)

Problem 3 (15 points): Consider a fixed \( m \times m \) matrix \( A \). Derive the condition number \( \kappa(x) \) for the following problem

Input: \( x \in \mathbb{C}^m \), Output: \( Ax \).

Note: You need to show all your derivations of your results.

Problem 4 (20 points) Rank Deficient Least Square Problems: Let \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \), and let \( r = \text{rank}(A) < n \), and write the SVD of \( A \) as

\[
A = [U_1, U_2] \begin{bmatrix}
\Sigma_1 & 0 \\
0 & 0
\end{bmatrix} [V_1, V_2]^T = U_1 \Sigma_1 V_1^T,
\]

where \( \Sigma_1 \) is \( r \times r \) nonsingular and \( U_1 \) and \( V_1 \) have \( r \) columns. Let \( \sigma = \sigma_{\text{min}}(\Sigma_1) \), the smallest nonzero singular value of \( A \). Consider the following rank deficient least square problem, for some \( b \in \mathbb{R}^m \),

\[
\min_{x \in \mathbb{R}^n} \|Ax - b\|_2.
\]

Show that:
1. all solutions $x$ can be written as $x = V_1 \Sigma_1^{-1} U_1^T b + V_2 z$, with $z$ an arbitrary vector;
2. the solution $x$ has minimal norm $\|x\|_2$ precisely when $z = 0$, and in which case, $|x|_2 \leq \|b\|_2 / \sigma$.

**Problem 5 (10 points)** Let $A \in \mathbb{C}^{10 \times 10}$ be nonsingular, and suppose that $\|A\|_2 = 100$ and $\kappa(A) = 100$.

1. What is $\|A^{-1}\|_2$?
2. Give (and justify) a sharp lower bound and a sharp upper bound for $\|A\|_F$. Find examples that show the bounds are sharp, i.e. that the lower and upper bounds are reached.

**Problem 6 (10 points)** The Moore-Penrose pseudoinverse $A^+$ for a possibly rank deficient matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ is defined as follows. Let $A = U \Sigma V^\top = U_1 \Sigma_1 V_1^\top$ as in Problem 4. Then $A^+$ is defined as $A^+ = V_1 \Sigma_1^{-1} U_1^\top$. Show that $A$ and $A^+$ satisfy the following identities.

1. $AA^+ A = A$;
2. $A^+ A A^+ = A^+$;
3. $A A^+ = (A A^+) \top$.

**Problem 7 (15 points)** Let $\{a_1, a_2, \ldots, a_m\}$ be a basis of $\mathbb{R}^m$. Then every vector $x \in \mathbb{R}^m$ can be written as a linear combination of the set $\{a_k\}$ that takes the following form

$$x = \sum_{k=1}^{m} (x, B a_k) a_k,$$

where $(x, y) = x^\top y$ is the inner product.

1. Let $A = [a_1 \mid a_2 \mid \cdots \mid a_m]$. Express $B$ in terms of $A$.
2. For any $n \leq m$, show that the operator

$$P : x \mapsto \sum_{k=1}^{n} (x, B a_k) a_k,$$

is a projector. (Hint: show that $(a_j, B a_k) = 0$ for $j \neq k$ and $= 1$ for $j = k$.)