A note on degrees and radians

J. David Taylor

October 6, 2016

Let $D = \mathbb{R} \cdot 1^\circ$ be the real vector space spanned by the unit "one degree". I will use capital Sine and Cosine to represent the trigonometric functions with degree arguments, and lower case sine and cosine to represent the trigonometric functions with real arguments. We have two maps

$$Exp : D \to S^1 : \alpha \mapsto (\mathrm{Cos}(\alpha), \mathrm{Sin}(\alpha))$$

and

$$exp : \mathbb{R} \to S^1 : x \mapsto (\cos(x), \sin(x)).$$

Since $\mathbb{R}$ is the universal cover of S^1 and $Exp(0) = exp(0)$, there is a unique continuous lift of $Exp : D \to S^1$ to a map $\phi : D \to \mathbb{R}$ that makes the diagram commute:

$$\begin{array}{ccc} D & \xrightarrow{\phi} & \mathbb{R} \\ & \overset{Exp}{\searrow} & \Big\downarrow exp \\ & & S^1 \end{array}$$

It is easy to see that ϕ is bijective and locally a homeomorphism. Therefore, ϕ is a homeomorphism. Furthermore, the relations on S^1 show that ϕ is locally additive. An inductive argument shows that ϕ is an isomorphism of abelian groups. Furthermore, continuity shows that ϕ is $\mathbb{R}$-linear. So there is a unique isomorphism of $\mathbb{R}$-vector spaces between D and R that commutes with the projections to S^1. Under this isomorphism, $\phi(360^\circ) = 2\pi$. Since ϕ is $\mathbb{R}$-linear, $\phi\left(\frac{180}{\pi}^\circ\right) = 1$.

Since ϕ is bijective and $\mathbb{R}$-linear, we can use this identification to get a field structure on D via the multiplication $\alpha \cdot_D \beta := \phi(\alpha) \cdot \beta$. Then we have

$$\phi(\alpha \cdot_D \beta) = \phi(\phi(\alpha)\beta) = \phi(\alpha)\phi(\beta).$$

So we may confirm all the field axioms are satisfied in D because they are in $\mathbb{R}$. Since there are no non-trivial automorphisms of $\mathbb{R}$ as a field, this is the unique way of extending the vector space D to a field isomorphic to $\mathbb{R}$. So D is isomorphic to R as a field in a unique way, given the restriction that the map commute with the exponential maps. Furthermore, $\frac{180}{\pi}^\circ$ is the multiplicative identity in D.

Since there are multiple constructions of the real numbers, but we identify them because there are unique isomorphisms between them, I see no reason to distinguish between D and $\mathbb{R}$, just as I see no reason to distinguish between the Dedekind Cut and Cauchy Sequence Constructions of the reals.