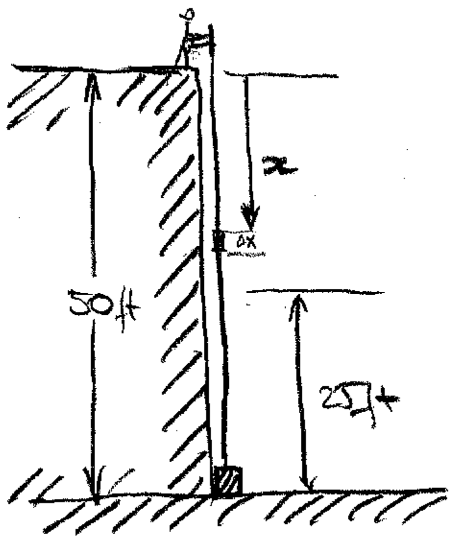
Exam 3, Math 129, Summer I 2008
Solutions

1. (10) A worker is on a roof of a 50 ft tall building. He needs to lift a 100 lb bucket of cement from the ground to a point 25 ft above the ground by pulling on a rope weighing 1 lb/ft. How much work is required? Carefully define your variable of integration.



Method 1: Rope is "cut" in little pieces, each piece being lifted a variable distance.

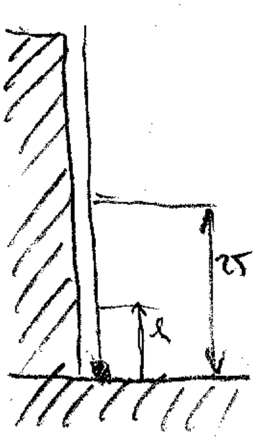
Let x denote the distance to the top of the building.

$$\text{WORK} = \text{work on bucket} + \left(\begin{array}{c}\text{work on the first}\\ \text{25 ft of rope}\end{array}\right) + \left(\begin{array}{c}\text{work on the top}\\ \text{25 ft of rope}\end{array}\right)$$

$$= \underbrace{100}_{\text{weight}} \cdot \underbrace{25}_{\substack{\text{distance}\\ \text{lifted}}} + \underbrace{25}_{\text{weight}} \cdot \underbrace{25}_{\substack{\text{distance}\\ \text{lifted}}} + \int_0^{25} \underbrace{1 \cdot \overbrace{x}^{\text{distance}} \cdot dx}_{\substack{\text{weight}\\ \text{of a little piece of rope}}}$$

$$= \frac{6875}{2} \text{ ft-lb}$$

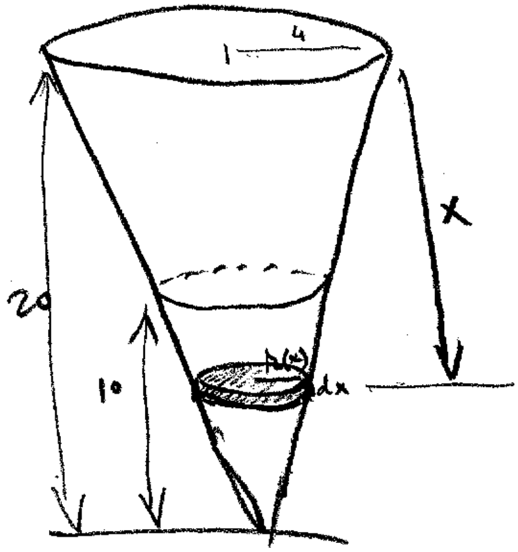
Method 2 (by Margaret Preston) Variable force, as in the example with the Hooke's law. Let h be the height above the ground.



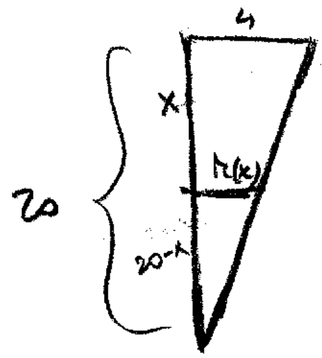
$$\text{Work} = \int_0^{25} \underbrace{(100 + 1\cdot(50-h))}_{\substack{\text{weight when}\\ h \text{ ft of the rope}\\ \text{were lifted}}} \cdot \underbrace{dh}_{\text{distance}} = \frac{6875}{2}$$

$$\boxed{\text{Work} = 6875/2} \text{ ft-lb}$$

2. (15) A water tank is in shape of right circular cone with height 20 ft and radius 4 ft at the top. Find the work required to pump all the water over the top of the tank, if the tank is filled with water to a depth of 10 ft. Recall that one cubic foot of water weighs 62.4 lb. Carefully define your variable of integration.



Let x be the distance to the top of the tank. We decompose the volume of water in horizontal slices. These are cylinders of height Δx and radius $r(x)$. We obtain $r(x)$ from a similar triangle:



$$\frac{r(x)}{4} = \frac{20-x}{20}$$

$$r(x) = \frac{20-x}{5}$$

$$\text{Work} = \sum \text{weight} \cdot \text{distance} = \int_{10}^{20} \underbrace{62.4}_{\text{density}} \cdot \underbrace{\pi \frac{1}{25}(20-x)^2}_{\text{volume of slice}} \cdot \overbrace{x}^{\text{distance}}\, dx$$

$$= \frac{62.4}{25}\pi \int_{10}^{20} \left[20(20-x)^2 - (20-x)^3\right] dx$$

$$= \frac{62.4}{25}\pi \left[-20\frac{(20-x)^2}{3} + \frac{(20-x)^4}{4}\right]\Bigg|_{10}^{20} = 10{,}400\pi \text{ ft-lb}$$

$$\boxed{\text{Work} = 10{,}400\pi \text{ ft-lb}}$$

3. (10) According with several studies, aspirin reduces the risk of a heart attack. You decide to take one tablet a day, each tablet containing 350 milligram of aspirin. The body eliminates the aspirin such that at the end of 24 hours about 0.5% of the initial amount is still left in the body.

(a) Let Q_n designate the quantity, in milligrams, of aspirin in the body right after the n^{th} tablet. Find Q_1, Q_2, Q_3, Q_4, and the general formula for Q_n.

Let $r = 0.005$. We have

$Q_1 = 350$

$Q_2 = 0.005\,Q_1 + 350 = (r+1)\cdot 350$

$Q_3 = 0.005\cdot Q_2 + 350 = (r^2+r+1)\cdot 350$

$Q_4 = 0.005\cdot Q_3 + 350 = (r^3+r^2+r+1)\cdot 350$

$$\boxed{Q_n = (r^{n-1}+\dots+r+1)\cdot 350 = \left(\frac{1-r^n}{1-r}\right)\cdot 350}$$

(b) What is $\lim_{n\to\infty} Q_n$?

$$\boxed{\lim_{n\to\infty} Q_n = \frac{1}{1-r}\,350 = 351.759}$$

(c) Let P_n designate the quantity, in milligrams, of aspirin in the body at the end of the n^{th} day, right before the next tablet is taken. Find P_1, P_2, P_3, P_4, and the general formula for P_n.

$P_1 = 0.005\cdot 350 = r\cdot 350$

$P_2 = 0.005\cdot Q_2 = r(r+1)\,350$

$P_3 = 0.005\,Q_3 = r\cdot(r^2+r+1)\,350$

$P_4 = 0.005\cdot Q_4 = r(r^3+r^2+r+1)\,350$

$$\boxed{P_n = r\cdot\frac{1-r^n}{1-r}\cdot 350}$$

4. (16) Use the integral test to decide whether the series below converge or diverge. In the case of convergence, provide an upper bound for the sum.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$

Method 1 (by Kane Peterson) Apply directly the integral test:

$$\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx = 2\arctan(\sqrt{x})\Big|_1^{\infty} = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \pi/2$$

So the series converges with upper bound $\pi/2$.

Method 2: We first use comparison $\frac{1}{n^{3/2}+n^{1/2}} < \frac{1}{n^{3/2}}$, then integral test:

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} = 1 + \int_1^{\infty} \frac{1}{x^{3/2}} dx = 1 - 2\frac{1}{x^{1/2}}\Big|_1^{\infty} = 3$$

Convergent ✓ Divergent ______ Bound= $\pi/2$

(b) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

Because $\int_2^{\infty} \frac{1}{x \ln x} dx = \ln(\ln x)\Big|_2^{\infty} = \infty$

the series itself is divergent

Convergent ______ Divergent ✓ Bound= ______

5. (10) Is the series $\overbrace{\frac{2}{3}}^{a_2} - \overbrace{\frac{3}{8}}^{a_3} + \overbrace{\frac{4}{15}}^{a_4} - \overbrace{\frac{5}{24}}^{a_5} + \frac{6}{35} - \frac{7}{48} + \dots$

absolutely convergent, conditionally convergent, or divergent? Explain your answer.

Solution: The series is alternating $a_2 - a_3 + a_4 - \dots = \sum_{n=2}^{\infty} (-1)^n a_n$,

with $a_n = \frac{n}{n^2-1}$

We have: • $a_n > a_{n+1}$ $\left(\Leftrightarrow \frac{n}{n^2-1} > \frac{n+1}{(n+1)^2-1} \Leftrightarrow n(n^2+2n) > (n^2-1)(n+1) \right.$

$\Leftrightarrow \cancel{n^3} + 2n^2 > \cancel{n^3} + n^2 - n - 1$

$\left. \Leftrightarrow n^2+n+1 > 0 \right)$

•• $\lim_{n\to\infty} a_n = 0$

Consequently the alternating series test applies and the given series is <u>convergent</u>.

For the absolute values series, $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$ we apply The integral test

$$\int_2^{\infty} \frac{x}{x^2-1}\,dx = \frac{1}{2} \ln(x^2-1)\Big|_2^{\infty} = \infty$$

We conclude that the absolute values series <u>diverges</u>.

Absolutely convergent ______ Conditionally convergent ✓ Divergent ______

6. (15) Determine whether the series below converge or diverge. State which convergence test you used.

(a) $\sum_{n=4}^{\infty} \frac{\sqrt{n-3}}{n+1}$

Let $a_n = \frac{\sqrt{n-3}}{n+1}$ and $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n-3}\cdot\sqrt{n}}{n+1} = \lim_{n\to\infty} \sqrt{\frac{n(n-3)}{(n+1)^2}} = 1$$

The limit comparison test applies.

Convergent ______ Divergent ✓ Test limit comparison

(b) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ Let $a_n = \cos(\frac{1}{n})$

$$\lim_{n\to\infty} \cos(\tfrac{1}{n}) = \cos(0) = 1$$

The general term does not go to zero, so series diverges.

Convergent ______ Divergent ✓ Test "go to ϕ test"

(c) $\sum_{n=1}^{\infty} \frac{\pi e^n - \sqrt{2}}{4^n}$ $= \pi \sum_{n=1}^{\infty} \left(\frac{e}{4}\right)^n - \sqrt{2} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$

$$= \pi \cdot \frac{\frac{e}{4}}{1-\frac{e}{4}} - \sqrt{2}\,\frac{\frac{1}{4}}{1-\frac{1}{4}}$$

Convergent ✓ Divergent ______ Test algebraic manipulations + geometric series test

7. (16) Find the radius of convergence and the interval of convergence for the series given below. Do check the ends of the interval as well.

(a) $\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}(x-\pi)^{2k}$

We consider the series as $\sum_{k=0}^{\infty} a_k$, where $a_k = (-1)^k \frac{1}{(2k)!}(x-\pi)^{2k}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{(2k+2)!}(x-\pi)^{2k+2} \cdot \frac{(2k)!}{(x-\pi)^{2k}} = \frac{(x-\pi)^2}{(2k+1)(2k+2)} \xrightarrow{k\to\infty} 0 < 1$$

for all x

Consequently, the Ratio test implies that $R = \infty$

R= ∞ Interval of convergence= $(-\infty, \infty)$

(b) $(x+1) + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{4}(x+1)^4 + \ldots$

Here the series is $\sum_{n=1}^{\infty} c_n (x+1)^n$, where $c_n = \frac{1}{n}$

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|} = \frac{1}{\lim_{n\to\infty}\frac{n}{n+1}} = 1$$

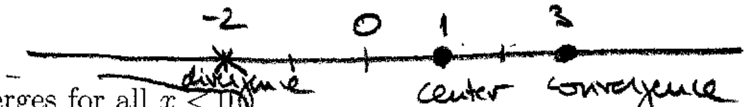
At $x=0$, series becomes $\sum_{n=1}^{\infty}\frac{1}{n}$, so divergent (harmonic)

At $x=-2$, series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, so convergent (oscillating harmonic)

R= 1 Interval of convergence= $[-2, 0)$

8. (8) Suppose that the power series $\sum_{n=0}^{\infty} C_n (x-1)^n$ converges for $x = 3$ and diverges for $x = -2$. Which of the following are True (**T**), False (**F**), or impossible (**I**) to determine. Briefly justify your answer.

The key is the next picture:



(a) **T** **F** **I** The power series diverges for all $x < 0$.

For all x in $(-1,0)$ we are guaranteed convergence.

(b) **T** **F** **I** The power series converges for $x = 3.5$.

3.5 is in the "gray area". We do not have enough information to decide what R is, so knowing that $2 \leq R \leq 3$ does not help answering this.

(c) **T** **F** **I** The power series converges for $x = 0$.

0 belongs to $(-1,3)$

(d) **T** **F** **I** The power series diverge for all $x > 4$.

This is guaranteed divergence.

This means that the radius of convergence must satisfy

$$2 \leq R \leq 3$$

with guaranteed convergence in $(-1,3)$ and guaranteed divergence outside $[-2,4]$.

All these follow from the properties of the convergence of power series.