Exam 2, Math 322, Summer 2008
Solutions

1. (10) Two solutions of the differential equation $x^2y'' - x(x+2)y' + (x+2)y = 0$, for $x > 0$, are

$$y_1(x) = x \quad \text{and} \quad y_2(x) = xe^x.$$

(a) (6 pts) Find the Wronskian $W(y_1, y_2)$.

$$W(y_1,y_2)(x) = \begin{vmatrix} x & xe^x \\ 1 & (1+x)e^x \end{vmatrix} = x^2e^x$$

$W(y_1, y_2) = \boxed{x^2e^x}$

(b) (4 pts) Are y_1 and y_2 linearly independent? Justify your answer.

Method 1
We have $W(y_1,y_2)(x) \neq 0$ on $(0,\infty)$, consequently the two functions are linearly independent.

Method 2: If y_1 & y_2 were linearly dependent, there would be nonzero constants c_1 and c_2 such that:

$c_1x + c_2xe^x = 0 \quad |:x \Rightarrow c_1 + c_2e^x = 0 \Rightarrow$

Yes/No: Yes

$\Rightarrow e^x = -\frac{c_1}{c_2}$ for all x

This is obviously impossible, so y_1 & y_2 must be independent.

2. (20) Consider one the Lagrange's equation with $n = 2$

$$(1-x^2)y'' - 2xy' + 6y = 0, \quad -1 < x < 1. \qquad (\star)$$

(a) (7 pts) Show that the coefficients a_m's of the power series expansion $y(x) = \sum_{m=0}^{\infty} a_m x^m$ of the general solution of Lagrange's equation satisfy the recursion relation:

$$a_{s+2} = \frac{(s+3)(s-2)}{(s+2)(s+1)} a_s, \quad \text{for } s \geq 2.$$

Introducing the series expansions of y, y', and y'' in the equation $(\star)$ we obtain:

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x \cdot \sum_{m=1}^{\infty} m a_m x^{m-1} + 6\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} 6a_m x^m = 0$$

let $m-2=s$ $\qquad$ $m=s$, but we have to start at $m=s=2$

$$\underbrace{2 \cdot a_2}_{s=0} + \underbrace{3 \cdot 2 a_3 x}_{s=1} + \sum_{s=2}^{\infty}\left[(s+2)(s+1)a_{s+2} - (s(s-1)+2s-6)a_s\right]x^s$$

$$- 2a_1 x + 6a_0 + 6a_1 x = 0$$

So the coefficient of x^s must vanish:

$$(s+2)(s+1)a_{s+2} = (s^2+s-6)a_s$$

. This is exactly the given identity.

(b) (5 pts) What are a_0, a_1, a_2, a_3, and a_4?

We also must have the coefficient of x vanish and the free term vanish:

#s $\quad 2a_2 + 6a_0 = 0 \quad \longrightarrow \quad a_2 = -3a_0$

x $\quad 6a_3 + 4a_1 = 0 \quad \longrightarrow \quad a_3 = -\frac{2}{3}a_1$

$a_0 =$ a_0 $\quad a_1 =$ a_1 $\quad a_2 =$ $-3a_0$ $\quad a_3 =$ $-\frac{2}{3}a_1$ $\quad a_4 =$ $\boxed{0}$ from the recursion relation

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(c) (4 pts) Use the recurrence relation from part (a) to find the next 4 terms of the power series expansion.

s=3 in the recursion relation gives: $a_5 = \frac{6\cdot 1}{5\cdot 4}\cdot a_3 = \frac{6}{5\cdot 4}\cdot\left(-\frac{2}{3}a_1\right) = -\frac{1}{5}a_1$

s=5 in the recursion relation gives: $a_7 = \frac{\cancel{8}\cdot\cancel{3}}{7\cdot\cancel{6}}\cdot a_5 = \frac{4}{7}\cdot\left(-\frac{1}{5}\right)a_1 = -\frac{4}{35}a_1$

$a_5 = -\frac{1}{5}a_1$ $a_6 = 0$ $a_7 = -\frac{4}{35}a_1$ $a_8 = 0$

(d) (4 pts) Use the results from previous parts to write the polynomial $P_2(x)$ that is a solution of ($\star$).

$$\boxed{P_2(x) = a_0\left(1 - 3x^2\right)}$$

Side Note: Remember that Legendre's polynomials are normalized such that $P_2(1) = 1$. In this case this requires that $\boxed{a_0 = -\frac{1}{2}}$

3. (20) Consider the following problem (it can be transformed into a Sturm-Liouville problem):

$$\textbf{(ODE)}\ y'' - 2y' + (1+\lambda)y = 0, \quad \text{on } [0,1], \qquad \textbf{(BC)}\ y(0) = 0, \quad y'(1) = 0.$$

By an eigenvalue of the above problem we understand a value of λ for which non-zero solutions of ((ODE)+(BC)) exist. Find an equation that the **positive** eigenvalues have to satisfy. How many positive eigenvalues are there?

The characteristic equation is:

$$r^2 - 2r + (1+\lambda) = 0$$

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 4(1+\lambda)}}{2} \underset{\lambda = \nu^2}{\overset{\text{let}}{=}} 1 \pm \nu i$$

So the general solution is $y(x) = c_1 e^x \cos(\nu x) + c_2 e^x \sin(\nu x)$.

We have:

$$0 = y(0) = c_1 + 0 \Rightarrow \boxed{c_1 = 0}$$

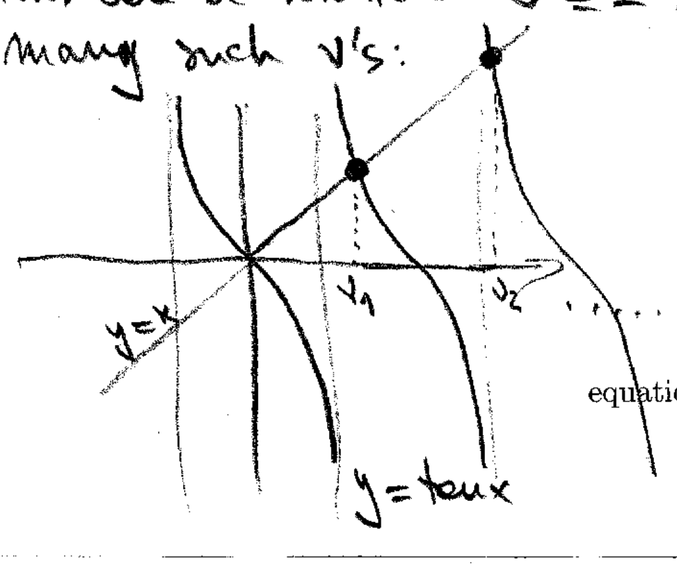
So $y(x) = c_2 e^x \sin(\nu x)$

$$y'(x) = c_2 e^x \left[\sin(\nu x) + \nu \cos(\nu x)\right]$$

$$0 = y'(1) = c_2 \cdot e \cdot \left[\sin(\nu) + \nu \cos \nu\right]$$

As we want $c_2 \neq 0$ it must be that $\boxed{\sin \nu + \nu \cos \nu = 0}$

This can be written $\nu = -\tan \nu$ and there are infinitely many such ν's:



$\lambda = \nu^2$ where ν satisfies

equation: $\boxed{\sin \nu + \nu \cos \nu = 0}$

4. (15) For two functions f and g on $[-\pi, \pi]$ we defined their **inner product** by

$$(f,g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

(a) (10pts) We also defined the **norm** of a function f by $\|f\| = \sqrt{(f,f)}$. What is the norm of

$$f(x) = \frac{1}{\sqrt{\pi}}\cos(nx), \text{ for } n = 1, 2, 3, \ldots?$$

(You may want to recall the double angle formula $\cos(2\alpha) = 2\cos^2(\alpha) - 1$.)

$$(f,f) = \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{\pi}}\cos(nx)\right)\cdot\left(\frac{1}{\sqrt{\pi}}\cos(nx)\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(2nx)+1}{2}\,dx$$

$$= \frac{1}{\pi}\,\frac{1}{2}\cdot 2\pi = 1$$

So $\boxed{\|f\| = \sqrt{(f,f)} = \sqrt{1} = 1}$

(b) (5 pts) What does it mean when we say that the set

$$\mathcal{B} = \left\{\frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\cos(2x), \frac{1}{\sqrt{\pi}}\cos(3x), \ldots, \frac{1}{\sqrt{\pi}}\cos(nx)\ldots\right\}$$

is an orthonormal family?

"ortho" = pairwise orthogonal functions, meaning that for $n \neq m$ $\left(\frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\cos(mx)\right) = 0$

"normal" = all functions have norm 1

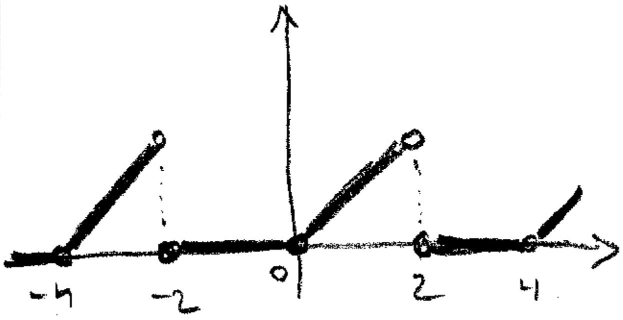
$\left\|\frac{1}{\sqrt{\pi}}\cos(nx)\right\| = 1.$ (You just proved this in part (a))

5. (20)

(a) (16pts) Find the Fourier series of the function:

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}.$$

The function is extended by periodicity with period $p = 4$.



$$p = 2L \Rightarrow L = 2$$

$$a_0 = \frac{1}{4}\int_0^2 x\,dx = \boxed{\frac{1}{2}}$$

$$a_n = \frac{1}{2}\int_0^2 x\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\left[\frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2\right]$$

$$= \boxed{\frac{2}{n^2\pi^2}\left((-1)^n - 1\right)}$$

$$b_n = \frac{1}{2}\int_0^2 x\sin\left(\frac{n\pi}{2}x\right) = \boxed{-\frac{2}{n\pi}(-1)^n}$$

$$\mathcal{F}(f)(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

(b) (4 pts) What is the value of the Fouries series at $x = -2$?

$$\mathcal{F}(f)(-2) = \frac{f^-(-2) + f^+(-2)}{2} = \frac{2+0}{2} = \boxed{1}$$

6. (15) MULTIPLE CHOICE. Circle the best response. No partial credit will be given on this problem and you do not need to justify your answers.

(a) Consider $f(x) = \sin x$, for $0 < x < \pi$. The coefficient a_0 of the odd extension of f is ...

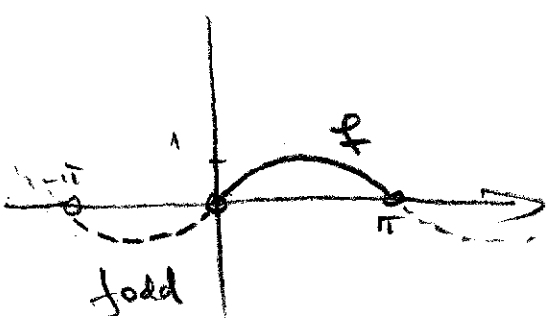
A. $-1/2$ **B.** $-1/4$ **C.** 0 **D.** $1/4$ **E.** $1/2$

The odd extension of f is an <u>odd function</u> (by definition).

The a_0 of any odd function is zero.

(b) Consider the same function as above $f(x) = \sin x$, for $0 < x < \pi$. The coefficient b_2 of the odd extension of f is ...

A. $-4/\pi$ **B.** 0 **C.** $1/\pi$ **D.** 1 **E.** $\pi/2$



f_{odd} is nothing but sine function

$$b_2 = \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\sin(x)}_{f_{odd}} \cdot \sin(2x)\,dx$$

$= 0$ because $\{\sin(nx) \mid n = 1, 2, \ldots\}$ is an orthogonal family.

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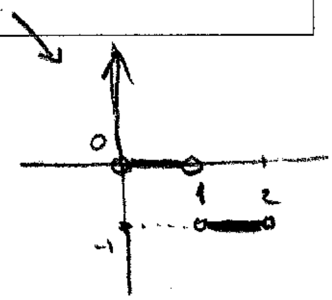
(c) Which of the following is an even extension to $-2 \le x \le 2$ of

$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases} \quad ?$$

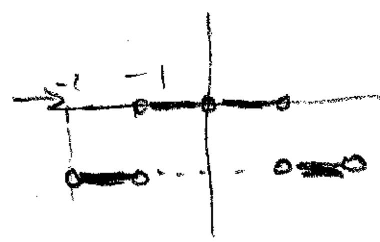


A.

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1 & -1 < x < 0 \\ 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

B.

$$f(x) = \begin{cases} -1 & -2 < x < -1 \\ 0 & -1 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$



C.

$$f(x) = \begin{cases} 0 & -2 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

D.

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ -1 & -1 < x < 0 \\ 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

E.

$$f(x) = \begin{cases} 1 & -2 < x < -1 \\ 0 & -1 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$