Name: Solution

Quiz #5, July 22, 2008

1. [5 pts] Show that $y(x) = cx - c^2$, with c constant, is solution of the differential equation $(y')^2 - xy' + y = 0$. This is called the *general solution*. Prove also that $y(x) = x^2/4$ is a solution. This is called *singular solution*.

Solution: This is a simple verification.

$y_1(x) = cx - c^2$, $y_1'(x) = c$, which introduced in the ODE gives: $c^2 - cx + cx - c^2 = 0$, which is true. So $y = cx - c^2$ is a solution. Similarly, $y_2(x) = x^2/4$ has $y_2'(x) = x/2$, which introduced in the ODE gives: $\frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0$, also true.

Method 1: The fundamental set corresponds to equal root of the characteristic equation $r_1 = r_2 = \sqrt{\pi}$. So the characteristic equation is $(r - \sqrt{\pi}) \cdot (r - \sqrt{\pi}) = 0$, or $r^2 - 2\sqrt{\pi}\, r + \pi = 0$. We obtain $\boxed{y'' - 2\sqrt{\pi}\, y' + \pi y = 0.}$

2. [5 pts] Find an ODE $y'' + ay' + by = 0$ that has the following basis for the space of solutions:

$$e^{x\sqrt{\pi}}, xe^{x\sqrt{\pi}}.$$

Method 2 (given by John Michael and Craig Oliver)

The general solution has form $y = c_1 e^{x\sqrt{\pi}} + c_2 x e^{x\sqrt{\pi}}$, with $y' = (c_1\sqrt{\pi} + c_2 + c_2\sqrt{\pi}x) e^{x\sqrt{\pi}}$, and $y'' = (c_1 \cdot \pi + 2c_2\sqrt{\pi} + c_2\pi x) e^{x\sqrt{\pi}}$.

Plugging these in the ODE gives:

$$\left[(c_1\pi + 2c_2\sqrt{\pi} + c_2\pi x) + a(c_1\sqrt{\pi} + c_2 + c_2\sqrt{\pi}x) + b(c_1 + c_2 x)\right] e^{x\sqrt{\pi}} = 0$$

After dividing with $e^{x\sqrt{\pi}}$, one obtains:

$$(\pi + a\sqrt{\pi} + b)\, c_2 x \; + \left((\pi + a\sqrt{\pi} + b) c_1 + (2\sqrt{\pi} + a) c_2 \right) = 0$$

So the coefficient of x has to be zero (assume $c_1, c_2 \neq 0$) and the free term has to be zero:

$$\begin{aligned} x:\;& \begin{cases} \pi + a\sqrt{\pi} + b = 0 \\ (\pi + a\sqrt{\pi} + b) c_1 + (2\sqrt{\pi} + a) c_2 = 0 \end{cases} \\ \#: & \end{aligned} \Rightarrow (2\sqrt{\pi} + a) c_2 = 0 \;\Big|\, : c_2$$

this just means that $\sqrt{\pi}$ is root of the characteristic equation

This shows that $\sqrt{\pi}$ is actually double root! $\longrightarrow$ $2\sqrt{\pi} + a = 0$

$$\boxed{a = -2\sqrt{\pi}}$$

This introduced in the first equation gives:

$$\pi + (-2\sqrt{\pi}) \cdot \sqrt{\pi} + b = 0$$

$$\boxed{b = \pi}.$$

This is the same answer as the one obtained with method 1, but it is at a deeper level. Morally with the above argument we proved that the form of the fundamental set of solutions determines unequivocally the 2nd order constant coefficients linear ODE.