

The N-body Problem

I. Vectors and Basic Operations

— *Vectors in \mathbb{R}^3 :*

(i) Let $\mathbf{u} = (x, y, z) \in \mathbb{R}^3$.

$$\text{(Norm) } |\mathbf{u}| = \sqrt{x^2 + y^2 + z^2}$$

$$\text{(Direction) } \mathbf{u}/|\mathbf{u}|.$$

(ii) Let $\mathbf{u}_1 = (x_1, y_1, z_1)$, $\mathbf{u}_2 = (x_2, y_2, z_2)$,

$$\mathbf{u}_1 \pm \mathbf{u}_2 = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2)$$

— *Inner product*

$$\mathbf{u}_1 \circ \mathbf{u}_2 = x_1x_2 + y_1y_2 + z_1z_2 = |\mathbf{u}_1||\mathbf{u}_2| \cos \theta,$$

if $\mathbf{u}_1 \circ \mathbf{u}_2 = 0$, then $\mathbf{u}_1 \perp \mathbf{u}_2$

— *Cross product:*

$\mathbf{u}_1 \times \mathbf{u}_2$ is perpendicular to the plane formed by \mathbf{u}_1 and \mathbf{u}_2 , obeying the right-hand rule,

$$|\mathbf{u}_1 \times \mathbf{u}_2| = |\mathbf{u}_1||\mathbf{u}_2| \sin \theta,$$

if $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{0}$, then $\mathbf{u}_1 \parallel \mathbf{u}_2$

— *Mixed Products:*

Let $\mathbf{u}_i = (x_i, y_i, z_i)$

(i) $(\mathbf{u}_1 \times \mathbf{u}_2) \circ \mathbf{u}_3 = (\mathbf{u}_2 \times \mathbf{u}_3) \circ \mathbf{u}_1 = (\mathbf{u}_3 \times \mathbf{u}_1) \circ \mathbf{u}_2$ is the volume of the parallelepiped formed by \mathbf{u}_1 , \mathbf{u}_2 and \mathbf{u}_3 .

(ii) $(\mathbf{u}_1 \times \mathbf{u}_2) \times \mathbf{u}_3 = (\mathbf{u}_1 \circ \mathbf{u}_3)\mathbf{u}_2 - (\mathbf{u}_2 \circ \mathbf{u}_3)\mathbf{u}_1$.

— *Computational Formulas:*

Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ and write

$$\mathbf{u}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}.$$

We have

$$\mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

and

$$(\mathbf{u}_1 \times \mathbf{u}_2) \circ \mathbf{u}_3 = \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

— *Concrete Examples:*

Let $\mathbf{u}_1 = (1, 2, 0)$, $\mathbf{u}_2 = (-2, 2, 3)$, $\mathbf{u}_3 = (0, 4, -1)$.

$$\mathbf{u}_1 + \mathbf{u}_2 = (-1, 4, 3), \quad \mathbf{u}_1 - \mathbf{u}_2 = (3, 0, -3)$$

$$\mathbf{u}_1 \circ \mathbf{u}_2 = -2 + 4 + 0 = 2$$

$$\mathbf{u}_1 \times \mathbf{u}_2 = (6, -3, 6).$$

II. The Two-body Problem

Let m_1, m_2 be two gravitational particles located at $\mathbf{r}_1, \mathbf{r}_2$ in \mathbb{R}^3 respectively. We derive a set of differential equations that describe the motions of these two particles.

— Three major elements:

(i) (The Law of Gravitation) The force from m_1 acting on m_2 is $\mathbf{F}_{12} = Gm_1m_2(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|^3$.

(ii) (The Calculus) The velocity of m_1 is $\mathbf{v}_1 = d\mathbf{r}_1/dt$; the acceleration of m_1 is $\mathbf{a}_1 = d\mathbf{v}_1/dt$.

(iii) (The Second Law of Mechanics) For the motion of an object, we have $\mathbf{F} = m\mathbf{a}$.

— Equation of motion

$$\frac{d^2\mathbf{r}_1}{dt^2} = Gm_2 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \quad \frac{d^2\mathbf{r}_2}{dt^2} = Gm_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

Let $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we have

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

where $\mu = G(m_1 + m_2)$.

We re-write this equation as

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = -\mu \frac{\mathbf{r}}{|\mathbf{r}|^3}.$$

— Integral of angular momentum:

$$\frac{d(\mathbf{r} \times \mathbf{v})}{dt} = 0$$

from which it follows that

$$\mathbf{r} \times \mathbf{v} = \mathbf{c} \tag{1}$$

— Integral of energy:

Let $r = |\mathbf{r}|$, we have

$$\mathbf{v} \circ \frac{d\mathbf{v}}{dt} = -\mu \frac{\mathbf{v} \circ \mathbf{r}}{r^3} = -\mu \frac{1}{r^3} \frac{d}{dt} \frac{1}{2} (\mathbf{r} \circ \mathbf{r}) = -\frac{\mu}{r^2} \frac{dr}{dt},$$

and it follows that

$$\frac{1}{2} |\mathbf{v}|^2 - \frac{\mu}{r} = h \quad (2)$$

— Integral of eccentricity:

$$\frac{d}{dt} \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{(\mathbf{r} \circ \mathbf{r}) \circ \mathbf{v} - (\mathbf{r} \circ \mathbf{v}) \mathbf{r}}{|\mathbf{r}|^3} = \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}}{|\mathbf{r}|^3}$$

We have

$$\frac{d}{dt} \frac{\mathbf{r}}{|\mathbf{r}|} = -\mu^{-1} \mathbf{c} \times \frac{d\mathbf{v}}{dt}$$

from which it follows

$$\mu \left(\frac{\mathbf{r}}{|\mathbf{r}|} + \mathbf{e} \right) = \mathbf{v} \times \mathbf{c} \quad (3)$$

where \mathbf{e} is a constant vector.

— We conclude

(i) From (1), \mathbf{r} is always in the plane that is perpendicular to a constant vector \mathbf{c} ;

(ii) From (3), \mathbf{e} is perpendicular to \mathbf{c} .

(iii) Let θ be the angle between \mathbf{r} and \mathbf{e} , we have from (3)

$$|\mathbf{r}| + |\mathbf{e}||\mathbf{r}| \cos \theta = \mu^{-1}|\mathbf{c}|^2$$

This is

$$|\mathbf{r}| = \frac{\mu^{-1}|\mathbf{c}|^2}{1 + |\mathbf{e}| \cos \theta}$$

(iv) Orbits for two-body problem:

- (a) Circular orbits if $|\mathbf{e}| = 0$,
- (b) Elliptic orbits if $0 < |\mathbf{e}| < 1$,
- (c) Parabolic if $|\mathbf{e}| = 1$,
- (d) Hyperbolic if $|\mathbf{e}| > 1$.

— From

$$\mu \left(\frac{\mathbf{r}}{|\mathbf{r}|} + \mathbf{e} \right) = \mathbf{v} \times \mathbf{c}$$

We have

$$\mu^2 \left(\frac{\mathbf{r}}{|\mathbf{r}|} + \mathbf{e} \right)^2 = |\mathbf{v}|^2 |\mathbf{c}|^2,$$

from which it follows that

$$\mu^2 \left(1 + 2 \frac{\mathbf{r} \circ \mathbf{e}}{|\mathbf{r}|} + |\mathbf{e}|^2 \right) = |\mathbf{v}|^2 |\mathbf{c}|^2.$$

Replace v^2 by $2h + 2\mu r^{-1}$ and $\mathbf{e} \circ \mathbf{r} = |\mathbf{c}|^2 \mu^{-1} - r$, we obtain

$$\mu^2 (|\mathbf{e}|^2 - 1) = 2h|\mathbf{c}|^2. \quad (4)$$

III. The N -body Problem

Let m_1, m_2, \dots, m_n be n gravitational particles located at $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$ in \mathbb{R}^3 respectively. We derive a set of differential equations that describe the motions of these particles.

— Three major elements:

(i) (The Law of Gravitation) The force from m_j acting on m_i is $\mathbf{F}_{ij} = Gm_i m_j (\mathbf{r}_j - \mathbf{r}_i) / |\mathbf{r}_j - \mathbf{r}_i|^3$.

(ii) (The Calculus) The velocity of m_i is $\mathbf{v}_i = d\mathbf{r}_i/dt$; the acceleration of m_i is $\mathbf{a}_i = d\mathbf{v}_i/dt$.

(iii) (The Second Law of Mechanics) For the motion of an object, we have $\mathbf{F} = m\mathbf{a}$.

– Equation of motion:

We have

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = \sum_{j \neq i} Gm_i m_j (\mathbf{r}_j - \mathbf{r}_i) / |\mathbf{r}_j - \mathbf{r}_i|^3$$

for $i = 1, \dots, n$.

— Law of Conservations:

(i) (Linear Momentum) $\sum_{i=1}^n m_i \mathbf{v}_i = M_0$

(ii) (Center of Mass) $\sum_{i=1}^n m_i \mathbf{r}_i = M_0 t + M_1$.

(iii) (Angular Momentum) $\sum_{i=1}^n m_i (\mathbf{r}_i \times \mathbf{v}_i) = \mathbf{c}$.

(iv) (Energy) $T - U = h$ where

$$T = \frac{1}{2} \sum_{i=1}^n m_i \mathbf{v}_i \circ \mathbf{v}_i, \quad U = \sum_{1 \leq i < j \leq n} G m_i m_j / |\mathbf{r}_i - \mathbf{r}_j|$$

IV. Homework

1. Let $\mathbf{u}_1 = (1, 2, 0)$, $\mathbf{u}_2 = (-2, 2, 3)$, $\mathbf{u}_3 = (0, 4, -1)$. Compute $(\mathbf{u}_1 \times \mathbf{u}_2) \circ \mathbf{u}_3$ and $(\mathbf{u}_1 \times \mathbf{u}_2) \times \mathbf{u}_3 = (-21, 6, 24)$.

2. For the two-body problem, show that if $|\mathbf{c}| \neq 0$, then the orbit is hyperbolic if $h > 0$, parabolic if $h = 0$ and elliptic if $h < 0$.

3. Show that if $|\mathbf{c}| = 0$, then the two-body moves on a line in \mathbb{R}^3 .
4. Find the minimum velocity to launch a space vehicle that escapes earth's gravity. (Hint: Use the integral of energy. You need to also look on the Web to find the value of the gravitational constant G , the mass of the Earth, and the Earth's radius.)
5. Find the minimum velocity to launch a space vehicle that escapes from the solar system. (Hint: Similar to the hint of previous problem. Look on the Web to find also the mass of the Sun, and the distance from the Sun to the Earth.)
6. Prove that n -body problem has no equilibrium solution.

7. Derive the integral of angular momentum and the integral of energy for the n -body problem.

8. We call a solution of the n -body problem a relative equilibrium if there exists a function $\phi(t) : \mathbb{R} \rightarrow \mathbb{R}$ and constant vectors $\mathbf{r}_1^{(0)}, \dots, \mathbf{r}_n^{(0)}$, so that $\mathbf{r}_1(t) = \phi(t)\mathbf{r}_1^{(0)}, \dots, \mathbf{r}_n(t) = \phi(t)\mathbf{r}_n^{(0)}$ is a solution of the n -body problem.

(i) Derive the equations $\phi(t)$ and $(\mathbf{r}_1^{(0)}, \dots, \mathbf{r}_n^{(0)})$ must satisfy for $\mathbf{r}_1(t) = \phi(t)\mathbf{r}_1^{(0)}, \dots, \mathbf{r}_n(t) = \phi(t)\mathbf{r}_n^{(0)}$ to be a relative equilibrium solution of the n -body problem.

(ii) Prove that the three-body problem has relative equilibrium solutions. (Hint: look into the case when the three bodies form an equilateral triangle).

9. Let $I = \frac{1}{2} \sum_{i=1}^n m_i \mathbf{r}_i \circ \mathbf{r}_i$, $M = \sum_{i=1}^n m_i$.

(i) Assume that the center of mass of the n -body system is at the origin of the physical space. (This is to say that $\sum_{i=1}^n m_i \mathbf{r}_i = \mathbf{0}$.) Verify that

$$I = \frac{1}{4M} \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} m_i m_j |\mathbf{r}_i - \mathbf{r}_j|^2.$$

(ii) Prove that for the solutions of the n -body problem, we have $\frac{d^2}{dt^2} I = 2T - U$ where T , U are as in the integral of energy.

(iii) Prove that, for a solution of the n -body problem, if the total energy $h > 0$, then the solution is unbounded in the sense that there exists no constant $K > 0$ so that $I(t) < K$ for all t .